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Gravitational collapse and formation of a black hole in a type II minimally modified gravity theory

Antonio De Felice,[a] **Kei-ichi Maeda,**[b,a] **Shinji Mukohyama**[a,c] **and Masroor C. Pookkillath**[a,d]

[a]Center for Gravitational Physics and Quantum Information,
Yukawa Institute for Theoretical Physics, Kyoto University,
606-8502, Kyoto, Japan
[b]Department of Physics, Waseda University,
Shinjuku, Tokyo 169-8555, Japan
[c]Kavli Institute for the Physics and Mathematics of the Universe (WPI),
The University of Tokyo, Kashiwa,
Chiba 277-8583, Japan
[d]Centre for Theoretical Physics and Natural Philosophy, Mahidol University,
Nakhonsawan Campus, Phayuha Khiri, Nakhonsawan 60130, Thailand

E-mail: antonio.defelice@yukawa.kyoto-u.ac.jp, maeda@waseda.jp, shinji.mukohyama@yukawa.kyoto-u.ac.jp, masroor.cha@mahidol.ac.th

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Abstract. We study the spherically symmetric collapse of a cloud of dust in VCDM, a class of gravitational theories with two local physical degrees of freedom. We find that the collapse corresponds to a particular foliation of the Oppenheimer-Snyder solution in general relativity (GR) which is endowed with a constant trace for the extrinsic curvature relative to the time t constant foliation. For this solution, we find that the final state of the collapse leads to a static configuration with the lapse function vanishing at a radius inside the apparent horizon. Such a point is reached in an infinite time-t interval, t being the cosmological time, i.e. the time of an observer located far away from the collapsing cloud. The presence of this vanishing lapse endpoint implies the necessity of a UV completion to describe the physics inside the resulting black hole. On the other hand, since the corresponding cosmic time t is infinite, VCDM can safely describe the whole history of the universe at large scales without knowledge of the unknown UV completion, despite the presence of the so-called shadowy mode.

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Contents

1 Introduction

It is a time in cosmology when we are awaiting for some answers to fundamental questions. What is gravity? The answer to this question might be still too far in time or even too difficult to be understood. The answer might in fact connect the quantum realm to the large scales of the universe. After all, even before aiming to find an answer to this question, currently we need to understand why cosmological data today seem to give a puzzling picture of the theory which is needed to model them.

In fact, on assuming all experiments being free from significant unknown systematics, it seems impossible to fit all the data at hand by means of ΛCDM. Therefore we are bound to explore alternative theories which may be giving instead a better fit to the data, fixing current cosmological tensions. In order to address this point, namely to have a theory which would allow general (but non vanishing) $H(z)$ and even a general (but positive) $G_{\rm eff}(z)/G_N$, the theories of VCDM and VCCDM were introduced [1, 2]. Here, by $G_{\rm eff}$ we mean the effective gravitational constant appearing in the modified Poisson equation which relates the over density perturbation of the matter, $\delta_{\rm m}$, to the Bardeen gravitational potential Ψ. On top of these interesting phenomenological properties, both VCDM and VCCDM only possess two degrees of freedom in the gravity sector, i.e. the two polarization modes of the tensorial gravitational waves. These two theories coincide with each other whenever cold dark matter components are negligible or not considered.

In the high k limit the VCDM theory acts like GR, i.e., $G_{\rm eff}/G_N \to 1$ (refer to [1]). Due to this fact VCDM theory cannot change $G_{\rm eff}(z)/G_N$ from unity, while VCCDM can change G_{cc}/G_N, where G_{cc} is the self gravitational coupling between cold dark matter particles. Therefore, VCDM and VCCDM will share the same solutions in the vacuum case (e.g. same BH solutions [3]) as well as solutions which only deal with baryonic matter (e.g. same star solutions [4]).

In both theories, we may find an appropriate cosmological model, which explains the observational data [2, 5]. In addition to it, when we focus on strongly gravitating compact objects, we have to reanalyze their viability. As we have shown in the previous papers, VCDM and VCCDM admit the Schwarzschild solution [3, 6] as well as a solution of TOV equation that is the same as the one in GR [4]. In VCDM/VCCDM, however, solutions with the same spacetime geometry but with different time slicings are physically different since the theory has a preferred time slicing. In order for a solution to be trustable, not only the spacetime geometry but also the time slicing should be regular. For example, the standard Schwarzschild-type foliation of a spherically-symmetric, static metric is ill-defined at the horizon and thus the corresponding solution in VCDM/VCCDM is physically singular while in GR it is simply a coordinate singularity. Then the question is how a black hole is formed in VCDM after gravitational collapse. In the formation of a black hole, a singularity should not appear on or outside the horizon. In this sense, any spherically-symmetric, static solution with the standard Schwarzschild-type foliation is no longer to be a valid black hole solution in VCDM after gravitational collapse. Hence, we ask, what kind of black hole solution in VCDM is found after gravitational collapse?

In this paper, in order to answer this question, we want to investigate the spherically symmetric collapsing solutions (or, at least, a subset of them) for a cloud of dust in VCDM (or for a cloud of baryonic dust in VCCDM). In a recent paper of ours [3], we have found that there is a subset of the solutions of VCDM theory which are also solutions of GR provided that the trace of the extrinsic curvature K (relative to the t-constant hypersurface) is a constant and that the VCDM auxiliary field ϕ is a constant as well. Motivated by the aforementioned result, we study the Oppenheimer-Snyder (OS) solution of GR rewritten in a coordinate patch which allows $K = K_\infty$, and study the matching conditions at the surface of the star for the metric field and all the auxiliary fields of VCDM.

VCDM and VCCDM are classified as Type-IIa Minimally Modified Gravity (MMG) theories [3, 7]. There are also other minimally modified gravity theories, each of which propagates only two local physical degrees of freedom [2, 8–20]. They break general four dimensional diffeomorphism invariance, because of the presence of a shadowy mode which defines a preferred foliation. For this preferred foliation the shadowy mode satisfies an elliptic equation of motion. Therefore different foliations mean physically different solutions in VCDM. For example in cosmology the trace of the extrinsic curvature K is a function of time and thus the theory is different from ΛCDM and has interesting phenomenology [5]. On the other hand, in the case of black holes, if we set K and ϕ to be constants then VCDM admits GR solutions. As a consequence, not all the slicings of the GR-OS solution will be appropriate

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(i.e. allowed by the equations of motion) slicings in VCDM. In particular the standard slicing of the OS solution provides an interior solution which is homogeneous and isotropic. In this case one finds the trace of the extrinsic curvature is merely proportional to the contraction rate $H(t)$ for the interior solution. But this means that at least this foliation cannot provide a solution of GR and that of VCDM at the same time. So one then wonders how a collapse in VCDM looks like.

In this paper we indeed show that a different foliation of the OS solution can describe a solution for both VCDM and GR. On following the results of [3], this VCDM-GR compatible foliation is defined as to make the trace of the extrinsic curvature for the solution a constant. This constant, after imposing the Israel junction conditions, needs to correspond to the trace of the extrinsic curvature for the outer metric, which then matches a constant de Sitter solution at infinity.

This leads to an interesting collapse solution in VCDM which then predicts the final state of the solution as reaching a radius (located inside the apparent horizon) where the lapse function vanishes (there and at any place up to the origin). This endpoint is reached in an infinite t-time interval, t being the time of a cosmological observer, i.e. an observer located far away from the collapsing cloud. The solutions, in this limit, reduces to the static solutions found in [21], with the difference that the free parameters of the solutions are fixed by the collapse dynamics.

This paper is organized as following. At first, we introduce the VCDM covariant action in section 2. Then, in section 3 we rewrite the GR (and VCDM) solutions in vacuum, corresponding to the outer metric patch, having the property that $K = K_\infty =$ constant. We find the properties of the collapse for the outer solutions in section 4. Instead, we study the interior solution and its collapse in section 5. We set the Israel junction conditions for VCDM in section 6. On fixing the appropriate boundary conditions at infinity, we finally study the collapse and its final point, which then leads asymptotically to static VCDM/GR solutions in section 7. We finally give our conclusions in section 8.

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2 The VCDM action

Here we show covariant action for the VCDM theory, which was first introduced in [3]:

$$
\begin{aligned}
S = M_{\rm P}^2 \int d^4x\sqrt{-g}\,\biggl\{&\frac{1}{2}\,R^{(4)} - V(\phi) - \frac{3}{4}\,\lambda^2 - \lambda\,(\nabla^\sigma \mathfrak{n}_\sigma + \phi) \\
&+ \frac{\lambda_2}{\mathcal{A}}\,[\gamma^{\tau\rho}\nabla_\tau\nabla_\rho\phi + \mathfrak{n}^\rho(\nabla_\rho\phi)\,\nabla^\sigma\mathfrak{n}_\sigma] + \lambda_T\,(1 + g^{\mu\nu}\mathfrak{n}_\mu\mathfrak{n}_\nu)\biggr\}\,, \qquad (2.1)
\end{aligned}
$$

$$\mathfrak{n}_\mu \equiv -\mathcal{A}\,\nabla_\mu \mathcal{T}\,, \qquad (2.2)$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + \mathfrak{n}^\mu\mathfrak{n}^\nu\,, \qquad (2.3)$$

where ϕ, $\mathcal{A}$ and $\mathcal{T}$ are 4D scalars. We have Lagrange multipliers λ, λ_2 and λ_T, which will determine the connections between the 4D scalars and the geometrical objects. Choosing the unitary gauge for the time coordinate, this action reduces to the equivalent action which was introduced in [1].

We can integrate out the field $\mathcal{A}$ by varying λ_T, and the action can then be rewritten as

$$S = M_{\rm P}^2 \int d^4x\sqrt{-g}\left\{\frac{1}{2}R^{(4)} - V(\phi) - \frac{3}{4}\lambda^2 - \lambda\left(\nabla^\sigma\mathfrak{n}_\sigma + \phi\right) + (-g^{\mu\nu}\nabla_\mu\mathcal{T}\,\nabla_\nu\mathcal{T})^{1/2}\,\lambda_2\left[\gamma^{\tau\rho}\nabla_\tau\nabla_\rho\phi + \mathfrak{n}^\rho(\nabla_\rho\phi)\,\nabla^\sigma\mathfrak{n}_\sigma\right]\right\}, \tag{2.4}$$

$$\mathfrak{n}_\mu = -(-g^{\mu\nu}\nabla_\mu\mathcal{T}\,\nabla_\nu\mathcal{T})^{-1/2}\,\nabla_\mu\mathcal{T}\,, \tag{2.5}$$

$$\gamma^{\mu\nu} = g^{\mu\nu} + \mathfrak{n}^\mu\mathfrak{n}^\nu\,, \tag{2.6}$$

so that $\nabla_\mu\mathcal{T}$ is timelike by construction. From this action we can find the covariant equations of motion, which were derived in appendix A of [3] (with a slightly different notation).

In what follows, using this VCDM model, we look for a spherically symmetric time-dependent solution which describes the gravitational dust collapse and the formation of a black hole. Note that the presented solution can also be applied to the VCCDM model.

3 The outer metric

In this section we write the Schwarzschild-de Sitter metric, i.e. the vacuum spherically symmetric GR solutions in the presence of a cosmological constant. It is given in such a way that the extrinsic curvature of the t-constant hypersurface has a constant trace. Then, as found in [3], automatically this solution will be also a solution of VCDM, provided that also the auxiliary field ϕ is constant.

3.1 Schwarzschild reloaded

Let us then assume an outer metric with the following ansatz

$$ds^2 = -\alpha(t,r)^2\left[1-\beta(t,r)^2/F(t,r)\right]\dot{T}^2\,dt^2 + 2\,\frac{\alpha\beta}{F}\,\dot{T}\,dt\,dr + \frac{dr^2}{F} + r^2 d\Omega^2\,, \tag{3.1}$$

$$d\Omega^2 = \frac{dz^2}{1-z^2} + (1-z^2)\,d\theta_2^2\,, \tag{3.2}$$

where we have defined $z = \cos\theta_1$, and have introduced a function $\dot{T} > 0$, explicitly showing time reparametrization invariance, which is a symmetry of VCDM. Then for this spherically symmetric ansatz we will consider the t-extrinsic curvature, that is the extrinsic curvature of the hypersurface $t = \text{const.}$ with normal vector $\mathfrak{n}_\alpha dx^\alpha = -\alpha\dot{T}dt$, with $\mathfrak{n}_\alpha\mathfrak{n}_\beta g^{\alpha\beta} = -1$, as

$$-K = \frac{2\beta}{r} + \frac{\beta\alpha_{,r}}{\alpha} + \beta_{,r} - \frac{\beta F_{,r}}{2F} + \frac{F_{,t}}{2F\alpha\dot{T}}\,. \tag{3.3}$$

We want to have a foliation of the spacetime, in which the trace of the extrinsic curvature takes the value

$$K = K_\infty = \text{constant}\,, \tag{3.4}$$

so that this solution belongs to the class of VCDM solutions which are also solutions of GR (provided that ϕ is also a constant).

We can formally solve the PDE in eq. (3.4), in terms of α, by finding

$$\alpha = \left\{W(t) - \frac{1}{\dot{T}}\int^r \frac{F_{,t}}{2F\beta}\exp\left[-\int^{r_2}\left(-\frac{K_\infty}{\beta} + \frac{F_{,r}}{2F} - \frac{\beta_{,r}}{\beta} - \frac{2}{r}\right)\mathrm{d}r_1\right]\mathrm{d}r_2\right\} \times \exp\left[\int^r\left(-\frac{K_\infty}{\beta} + \frac{F_{,r}}{2F} - \frac{\beta_{,r}}{\beta} - \frac{2}{r}\right)\mathrm{d}r_3\right]. \tag{3.5}$$

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Given the metric in eq. (3.1), we can find its own Einstein tensor, $G^\mu{}_\nu$. In fact, its (t,r) component can be found to be

$$G^t{}_r = -(K_\infty r + r\beta_{,r} + 2\beta)\,[\ldots]\,, \tag{3.6}$$

where the dots represent, in general, a non-zero quantity (which, if vanishing, would make α not well defined). Setting this Einstein tensor component, $G^t{}_r$, to vanish leads to

$$\beta = -\frac{1}{3}\,K_\infty r + \frac{\kappa(t)}{r^2}\,. \tag{3.7}$$

With the above solution for β we find the following relation

$$\exp\left[\int^r\left(-\frac{K_\infty}{\beta} + \frac{F_{,r}}{2F} - \frac{\beta_{,r}}{\beta} - \frac{2}{r}\right)\mathrm{d}r_1\right] = \exp\left[\int^r\left(\frac{F_{,r}}{2F}\right)\mathrm{d}r_1\right] = W_1(t)\,\sqrt{F}\,, \tag{3.8}$$

and then eq. (3.5) can be written as

$$\begin{aligned}\alpha &= \left(W(t)W_1(t) - \frac{W_1(t)}{\dot{T}W_2(t)}\int^r \frac{F_{,t}}{2F\beta\sqrt{F}}\mathrm{d}r_1\right)\sqrt{F}\\ &= \left(1 - \frac{1}{\dot{T}}\int^r \frac{F_{,t}}{2\beta F^{3/2}}\mathrm{d}r_1\right)\sqrt{F}\,,\end{aligned} \tag{3.9}$$

having fixed the integration functions, as to make $\alpha \to 1$ for $F_{,t} \to 0$ when $F \to 1$.

In cosmology in GR, on de Sitter with the standard flat chart coordinates, we have $K = K_\infty = 3H_\infty$. In this case the first Friedmann equation leads to $3M_{\mathrm{P}}^2H_\infty^2 = M_{\mathrm{P}}^2\Lambda_{\mathrm{eff}}$, or $\Lambda_{\mathrm{eff}} = 3H_\infty^2 = \frac{1}{3}\,K_\infty^2$. Then this fixes K_∞ in terms of the cosmological constant, or in terms of H_∞, for this foliation. Therefore let us now fix the Schwarzschild background to satisfy the Einstein equation as

$$G^\mu{}_\nu = -\Lambda_{\mathrm{eff}}\delta^\mu{}_\nu = -\frac{1}{3}\,K_\infty^2\,\delta^\mu{}_\nu. \tag{3.10}$$

If we solve the time-time component, that is $G^t{}_t = -\frac{1}{3}\,K_\infty^2$, then we find

$$F = 1 + \frac{Z(t)}{r} + \frac{\kappa(t)^2}{r^4}\,. \tag{3.11}$$

From the r-r component, that is $G^r{}_r = -\frac{1}{3}\,K_\infty^2$, we need to set the following constraint

$$(-2K_\infty\dot{\kappa} - 3\dot{Z})\,[\ldots] = 0\,, \tag{3.12}$$

where, once more, the dots represent, in general, a non-zero quantity. Then we can solve

$$Z(t) = -r_H - \frac{2}{3}\,K_\infty\,\kappa(t)\,, \tag{3.13}$$

where r_H is an integration constant. Then finally

$$F = 1 - \frac{r_H}{r} - \frac{2}{3}\frac{K_\infty\kappa(t)}{r} + \frac{\kappa(t)^2}{r^4}\,, \tag{3.14}$$

$$\beta = -\frac{1}{3}\,K_\infty r + \frac{\kappa(t)}{r^2}\,, \tag{3.15}$$

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so that

$$F_{,t} = \frac{2\dot{\kappa}}{r^2}\,\beta\,, \tag{3.16}$$

and

$$\alpha = \left(1 - \frac{\dot{\kappa}}{\dot{T}}\int_{r_0}^{r}\frac{1}{r_1^2 F^{3/2}}\mathrm{d}r_1\right)\sqrt{F}\,, \tag{3.17}$$

where r_0 is just an integration constant. The other equations of motion are also satisfied, so that the solution is determined (up to time reparametrization $\dot{T}$) by the free function $\kappa(t)$, and the constants K_∞, and r_0. In particular, we will set $r_0 \to \infty$ as to have $\lim_{r\to\infty}\alpha = 1$.

In summary, this is a solution which is nothing but the standard Schwarzschild de Sitter of GR in a constant mean curvature slicing, having a constant trace for the t-extrinsic curvature K, making this solution also a solution of VCDM [3] (provided also ϕ is a constant). Since this solution is also a solution of GR, it is not a surprise that this solution were already found in the literature in the context of maximal slicing or constant mean curvature slicing (see e.g. [22–26].) Note that this solution is a subset of the general time-dependent solutions found in VCDM. Indeed, there exist other time-dependent VCDM solutions which are not found in GR. However we study just this GR-type solution in this paper since we would like to show the existence of VCDM solution which describes gravitational collapse and formation of a black hole.

Although this solution holds true for any value of K_∞, we need to set $K_\infty = 3H_\infty = \frac{3}{2}\,V_{,\phi_\infty} - \phi_\infty$, the last relation holding true for VCDM, as to have a de Sitter limit at infinity.[1] Then its contribution, corresponding to an effective cosmological constant contribution, on astrophysical scales/configurations, can be safely neglected, so that when we deal with numerics we will set K_∞ to vanish.

3.2 Presence of the apparent horizons

We have just seen that a constant trace of the extrinsic curvature K, relative to the t-slicing hypersurface, corresponds to having

$$F(t,r) = 1 - \frac{r_H}{r} - \frac{2}{3}\frac{K_\infty\kappa(t)}{r} + \frac{\kappa(t)^2}{r^4}\,. \tag{3.18}$$

In the above expression we have a free time-dependent function $\kappa(t)$. Let us see how the freedom of $\kappa(t)$ affects (or not) the presence of the apparent horizon for the metric in eq. (3.1). For this, once more we consider a completely general spherically symmetric ansatz rewritten for our convenience as

$$ds^2 = -(N^2 - B^2)\,dt^2 + 2B\mathcal{F}dt\,dr + \mathcal{F}^2\,dr^2 + r^2 d\Omega^2\,. \tag{3.19}$$

In the following we consider $N > 0$, as well as $\mathcal{F}$ and B (a negative B would be regarded as a flip in $t \to -t$), and all of them are functions of t and r. Then let us consider two lightlike vectors

$$l^\mu\partial_\mu = \partial_t + \frac{N-B}{\mathcal{F}}\,\partial_r\,, \tag{3.20}$$

$$w^\mu\partial_\mu = \partial_t - \frac{N+B}{\mathcal{F}}\,\partial_r\,, \tag{3.21}$$

[1] Outside the de Sitter limit, it is not trivial to find a solution which can be extended up to infinity. This because in a universe with matter, before reaching r-infinity, matter sources (baryons, dark matter) will give non negligible contributions (sourcing furthermore a time dependence for the field ϕ) which lay outside the validity of these $\Lambda_{\rm eff}$-vacuum solutions.

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which satisfy

$$l^\mu l_\mu = 0 = w^\mu w_\mu \,, \qquad l^\mu w_\mu = -2N^2 \, < 0. \tag{3.22}$$

Using the above lightlike vectors, we can define an orthogonal projector

$$h^{\mu\nu}_{\rm lw} = g^{\mu\nu} + \frac{l^\mu w^\nu + l^\nu w^\mu}{(-l^\sigma w_\sigma)} \,, \tag{3.23}$$

so that $h^{\mu\nu}_{\rm lw} l_\nu = 0 = h^{\mu\nu}_{\rm lw} w_\nu$. Then we can build the following two scalars

$$\theta_l = h^{\mu\nu}_{\rm lw} \nabla_\nu l_\mu = \frac{2}{r\mathcal{F}} (N - B) \,, \qquad \theta_w = h^{\mu\nu}_{\rm lw} \nabla_\nu w_\mu = -\frac{2}{r\mathcal{F}} (N + B) \,. \tag{3.24}$$

Let us now consider the marginally outer trapped surface (MOTS), i.e. the apparent horizon, that is the surface $r = r_{AH}(t)$ for which

$$\theta_l(t, r = r_{AH}(t)) = 0 \,, \qquad \theta_w(t, r = r_{AH}(t)) < 0 \,. \tag{3.25}$$

In this case we have a non trivial solution of the form

$$N(t, r_{AH}(t)) = B(t, r_{AH}(t)) \,, \qquad \text{for which} \qquad g^{rr}(t, r = r_{AH}(t)) = \frac{N^2 - B^2}{N^2 F^2} = 0 \,. \tag{3.26}$$

So we have a MOTS, an apparent horizon at $r = r_{AH}(t)$. Let us see what happens for the metric under study. On identifying the metric in eq. (3.1) with the one in eq. (3.19), we have

$$\mathcal{F}^2 = \frac{1}{F} \,, \qquad \text{or} \qquad \mathcal{F} = F^{-1/2} \,, \tag{3.27}$$

$$B\mathcal{F} = \frac{\alpha\beta}{F} \dot{T} \,, \qquad \text{or} \qquad B = \frac{\alpha\beta\dot{T}}{\sqrt{F}} \,, \tag{3.28}$$

$$N^2 - B^2 = \alpha^2 \dot{T}^2 \left[1 - \beta^2/F\right] , \qquad \text{or} \qquad N = \alpha \dot{T} \,. \tag{3.29}$$

In this case we have the MOTS for a nonzero α and $\dot{T}$, if

$$\alpha\dot{T} = \frac{\alpha\beta\dot{T}}{\sqrt{F}} \,, \qquad \text{or} \qquad F = \beta^2 \,. \tag{3.30}$$

The above equation on using eq. (3.14) and eq. (3.15) leads to

$$1 - \frac{r_H}{r_{AH}} - \frac{2}{3}\frac{K_\infty \kappa(t)}{r_{AH}} + \frac{\kappa(t)^2}{r_{AH}^4} = \frac{1}{9} K_\infty^2 r_{AH}^2 + \frac{\kappa(t)^2}{r_{AH}^4} - \frac{2}{3}\frac{K_\infty \kappa(t)}{r_{AH}} \tag{3.31}$$

or

$$1 - \frac{r_H}{r_{AH}} - \frac{1}{9} K_\infty^2 r_{AH}^2 = 0 \,. \tag{3.32}$$

This shows that there are only two horizons, the event horizon and the cosmological horizon, and also that $\dot{r}_{AH} = 0$, independently of the function $\kappa(t)$. So the free function $\kappa(t)$ does not affect the position/behavior of the apparent horizon and we are still able to choose it the way we think it is more useful, at least locally. However, as we shall see later on, the value of $\kappa(t)$ will be determined by the dynamics of the collapse.

3.3 VCDM coordinate patch

We know that an exact GR configuration/solution is also shared by the VCDM theory, provided that the VCDM auxiliary field ϕ has a constant profile and that the trace of the extrinsic curvature tensor (for the t-foliation in VCDM) is also a constant. Then the previous GR solution can be embedded also in VCDM, since $K = K_\infty$. The vacuum spacetime has an apparent horizon at the zeros of the function $1 - \frac{r_H}{r} - \frac{1}{9} K_\infty^2 r^2$.

We want to show here that it is possible that, for some value of $\kappa(t)$ at a given time, the lapse function $N = \alpha \dot{T}$, for $\dot{T} > 0$, never vanishes for any $r > 0$. In particular, for a constant $\kappa(t) = \kappa_0$, we have that $\alpha = \sqrt{F}$, so that also F never vanishes for this coordinate patch. In any case, the metric would still be possessing an apparent horizon so that the singularity, which may appear at the center $r = 0$, is never naked (for $r_H > 0$ and $K_\infty^2 r_H^2 \ll 1$). For a non-vanishing F, this coordinate choice would be able to describe all the $r > 0$ region.

Hence, for simplicity, in the following we only consider a constant κ and look for the domain of κ's which makes F (or α) never vanish. The study of this coordinate patch will help us understanding the final state of the collapse in VCDM theory. In particular, let us try to set the equation $r^4 F = 0$, as to have double real roots and two complex ones. Let us try to find the "extremal" value of κ for which this happens.

Then in this case, on assuming $r > 0$, let us find constants $\xi_{1,2,3}$ to allow for this possibility to hold true. Then we have

$$r^4 F = r^4 - \left(r_H + \frac{2}{3} K_\infty \kappa\right) r^3 + \kappa^2 = \frac{1}{\xi_1} (r - r_1)^2 \left[\xi_1 r^2 + \xi_2 r + \xi_3\right], \tag{3.33}$$

with $\xi_2^2 - 4\xi_1\xi_3 < 0$ and $\xi_1 \neq 0$. The absence of terms linear in r in the r.h.s. of eq. (3.33) imposes that

$$\xi_3 = \frac{\xi_2 r_1}{2}\,. \tag{3.34}$$

Then the absence of quadratic terms in r also imposes that

$$\xi_2 = \frac{2\xi_1 r_1}{3}\,, \qquad \text{and} \qquad \xi_3 = \frac{\xi_1 r_1^2}{3}\,. \tag{3.35}$$

At this point we have that

$$\xi_2^2 - 4\xi_1\xi_3 = \left(\frac{2\xi_1 r_1}{3}\right)^2 - \frac{4\xi_1^2 r_1^2}{3} = -2\left(\frac{2\xi_1 r_1}{3}\right)^2 < 0\,, \tag{3.36}$$

and

$$r^4 - \left(r_H + \frac{2}{3} K_\infty \kappa\right) r^3 + \kappa^2 = r^4 - \frac{4 r_1 r^3}{3} + \frac{r_1^4}{3}\,. \tag{3.37}$$

For the last equation to make sense as an identity we need

$$r_1 = \frac{1}{4}\left(3 r_H + 2\kappa K_\infty\right), \tag{3.38}$$

$$\kappa^2 = \frac{r_1^4}{3} = \frac{1}{3}\left[\frac{1}{4}\left(3 r_H + 2\kappa K_\infty\right)\right]^4. \tag{3.39}$$

From the first equation we can derive the value of r_1, and we can see that for vanishing K_∞ we indeed find $r_1 = 3r_H/4$, as expected (see e.g. [6]). The second equation does not trivially

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hold, unless we find the values of κ for which the equation is satisfied. Therefore we find for positive κ's that

$$\kappa=\kappa_+\equiv\frac{9r_H^2}{-6K_\infty r_H+4\sqrt{3}\left(\sqrt{-2\sqrt{3}K_\infty r_H+4}+2\right)}=\frac{3}{16}\sqrt{3}r_H^2+\frac{9}{64}K_\infty r_H^3+\mathcal{O}[K_\infty^2 r_H^4]\,, \tag{3.40}$$

and for negative κ's that

$$\kappa=\kappa_-=-\frac{9r_H^2}{4\sqrt{3}\left(\sqrt{4+2\sqrt{3}K_\infty r_H}+2\right)+6K_\infty r_H}=-\frac{3}{16}\sqrt{3}r_H^2+\frac{9}{64}K_\infty r_H^3+\mathcal{O}[K_\infty^2 r_H^4]\,. \tag{3.41}$$

Hence, we find the following two cases (see the appendix for explicit calculations).

1. For $\kappa_-\le\kappa<0$ or $0<\kappa\le\kappa_+$, the coordinate patch can get inside the Schwarzschild radius but has a lower bound r_1, that is

$$0<\kappa\le\kappa_+ : r_{1,+}\le r_1<r_H\,, \tag{3.42}$$

$$\kappa_-\le\kappa<0:\quad r_{1,-}\le r_1<r_H\,, \tag{3.43}$$

where

$$r_{1,+}\equiv\frac{1}{4}(3r_H+2K_\infty\kappa_+)=-\frac{\sqrt{3}\sqrt{-2\sqrt{3}K_\infty r_H+4}}{2K_\infty}+\frac{\sqrt{3}}{K_\infty}\approx\frac{3r_H}{4}+\frac{3\sqrt{3}}{32}K_\infty r_H^2\,, \tag{3.44}$$

$$r_{1,-}\equiv\frac{1}{4}(3r_H+2K_\infty\kappa_-)=-\frac{\sqrt{3}}{K_\infty}+\frac{\sqrt{3}\sqrt{4+2\sqrt{3}K_\infty r_H}}{2K_\infty}\approx\frac{3r_H}{4}-\frac{3\sqrt{3}}{32}K_\infty r_H^2\,, \tag{3.45}$$

and the numbers $r_{1,\pm}$ have been expanded in terms of the small parameter $K_\infty r_H$. For $\kappa=0$, the solution cannot get inside the Schwarzschild radius and the lapse vanishes at $r=r_H$. At $r=r_1$, both the functions F and $\alpha=\sqrt{F}$ vanish, leading to a vanishing lapse function, in general.

2. For $\kappa<\kappa_-$ or $\kappa>\kappa_+$ the solutions of $r^4F=0$ are all complex and F never vanishes, so as α.

In any case, we stress once more that, although at this level, there is the freedom of the choice of κ, we will see that at the end of the collapse, κ will be set to belong to the first possibility, namely $\kappa\le\kappa_+$ (for positive κ's), as to exclude the second case.

4 Collapsing outside vacuum solution

Based on the previous results, in the following we will consider the outer vacuum solution endowed with a time-dependent κ. So we will consider then as the outside solution (i.e. the solution valid outside the collapsing star) the one given by

$$F=1-\frac{r_H}{r}-\frac{2}{3}\frac{K_\infty\kappa(t)}{r}+\frac{\kappa(t)^2}{r^4}\,, \tag{4.1}$$

$$\beta=-\frac{1}{3}K_\infty r+\frac{\kappa(t)}{r^2}\,, \tag{4.2}$$

$$\alpha=\left(1-\frac{\dot{\kappa}}{\dot{T}}\int_{r_0}^{r}\frac{1}{r_1^2F(t,r_1)^{3/2}}\,\mathrm{d}r_1\right)\sqrt{F}\,, \tag{4.3}$$

and for the sake of clarity we rewrite the metric as

$$
\begin{aligned}
ds^2 &= -\frac{\alpha^2}{F}\left[F-\beta^2\right]\dot{T}^2\,dt^2 + 2\,\frac{\alpha\beta\dot{T}}{F}\,dt\,dr + \frac{dr^2}{F} + r^2 d\Omega^2\,,\\
d\Omega^2 &= \frac{dz^2}{1-z^2} + (1-z^2)\,d\theta_2^2\,,
\end{aligned}
$$

where, once more, the function $\dot{T}(t)$ is due to time-reparametrization invariance. In the following we will also define the functions $g_{1,2}$ as

$$
\alpha(t,r) = g_2(t,r) - \frac{\dot{\kappa}}{\dot{T}}\,g_1(t,r)\,, \tag{4.4}
$$

$$
g_2(t,r) = \sqrt{F}\,, \tag{4.5}
$$

$$
g_1(t,r) = \sqrt{F}\int_{r_0}^{r}\frac{\mathrm{d}r_1}{r_1^2\left[1 - r_H/r - \frac{2}{3}\frac{K_\infty\kappa(t)}{r} + \kappa(t)^2/r^4\right]^{3/2}}\,. \tag{4.6}
$$

This implicit form is convenient in order not to hide any contribution coming from $\dot{T}$.

We will define the hypersurface for the external metric coordinates on which we will match the two metrics by

$$
\Phi(x^\mu) = r - R(t) = 0\,, \tag{4.7}
$$

where $R(t)$ is still a function of time to be determined, which corresponds to the radius of the collapsing cloud/star as seen from the outside metric point of view. Then the normal n_μ to this surface is proportional to the gradient of Φ, namely

$$
n_\mu dx^\mu \propto dr - \dot{R}\,dt\,, \tag{4.8}
$$

with the condition that $n_\alpha n^\alpha = 1$ for the normal vector to the hypersurface. Then we find

$$
n_\mu dx^\mu = \frac{1}{\sqrt{\Delta_n}}\,(g_2\dot{T} - g_1\dot{\kappa})\,(dr - \dot{R}\,dt)\,, \tag{4.9}
$$

$$
\Delta_n \equiv g_2^2\dot{T}^2\left(F-\beta^2\right) - 2g_2\dot{T}[\dot{\kappa}(F-\beta^2)g_1 + \beta\dot{R}] + g_1^2\dot{\kappa}^2\left(F-\beta^2\right) + 2\dot{R}g_1\beta\dot{\kappa} - \dot{R}^2\,. \tag{4.10}
$$

Now we can define the following projection tensors

$$
h_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta\,, \tag{4.11}
$$

$$
h_\alpha{}^\beta = h_{\alpha\mu}g^{\mu\beta}\,, \tag{4.12}
$$

so that

$$
h_\alpha{}^\beta n_\beta = 0\,, \qquad h_{\alpha\beta}n^\beta = 0. \tag{4.13}
$$

We can define then the extrinsic curvature for this hypersurface as

$$
\mathcal{K}_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma n_{\rho;\sigma}\,. \tag{4.14}
$$

This extrinsic curvature tensor should not be confused with the extrinsic curvature tensor $K_{\mu\nu}$ which was defined for the t-constant hypersurfaces. We further define three vectors as

$$
e^\mu{}_a = \frac{\partial x^\mu}{\partial y^a}\,, \tag{4.15}
$$

where y^a are coordinates on the hypersurface, here chosen to be (t, z, θ_2). Then we have

$$e^\mu{}_t\,\partial_\mu = \partial_t + \dot{R}\,\partial_r\,, \qquad e^\mu{}_z\,\partial_\mu = \partial_z\,, \qquad e^\mu{}_{\theta_2}\,\partial_\mu = \partial_{\theta_2}\,, \tag{4.16}$$

satisfying $n_\mu e^\mu{}_a = 0$, out of which we can find twelve scalars as

$$h_{ab} = h_{ba} = h_{\mu\nu} e^\mu{}_a e^\nu{}_b|_{r=R(t)}\,, \qquad \mathcal{K}_{ab} = \mathcal{K}_{ba} = \mathcal{K}_{\mu\nu} e^\mu{}_a e^\nu{}_b|_{r=R(t)}\,. \tag{4.17}$$

We impose the standard Israel junction conditions at the surface of the star, so that these expressions need to be continuous on the hypersurface joining the two different GR solutions. Out of these scalars, given the spherical symmetric ansatz, only the diagonal components do not vanish.

5 Collapsing inside dust solution

Let us now study the collapse from the point of view of the inside metric. We then write the inside metric as a spatially closed homogeneous and isotropic metric, however, by choosing a quite general time slicing, as follows

$$ds^2 = -a\big(f(t,\chi)\big)^2\,[f_{,t}dt + f_{,\chi}d\chi]^2 + a\big(f(t,\chi)\big)^2\left[\frac{d\chi^2}{1-\chi^2} + \chi^2 d\Omega^2\right], \tag{5.1}$$

where we have started with the conformal time η with $N(\eta) = a(\eta)$ and made the general coordinate transformation $\eta = f(t,\chi)$, that respects the spherical symmetry, to relate η to the VCDM time t. The coordinate transformation, i.e. the slicing is chosen as to have $K = K_\infty$ for the trace of the extrinsic curvature of the t-const hypersurface. In fact we want the same VCDM slicing to hold both outside and inside the hypersurface to accommodate the GR solution in VCDM, and for this purpose we need to require that K for the t = constant hypersurface be constant everywhere, so as the VCDM field ϕ. In GR, although this choice is allowed, the standard coordinates of the Oppenheimer-Snyder solution would be the simplest to implement. In VCDM, however, different slicings correspond to physically different solutions, so that we are looking for that particular slicing which satisfies $K = K_\infty$. For this metric, we have that the normal to the t = constant surface can be written as

$$\mathfrak{n}_\alpha dx^\alpha = -\frac{a f_{,t}}{\sqrt{1 - f_{,\chi}^2(1-\chi^2)}}\,dt\,, \tag{5.2}$$

which is well defined only if $1 - f_{,\chi}^2(1-\chi^2) > 0$. Out of this covector we can find its associate induced metric, $\mathfrak{h}_{\alpha\beta} = g_{\alpha\beta} + \mathfrak{n}_\alpha\mathfrak{n}_\beta$, its extrinsic curvature tensor, $K_{\alpha\beta} = \mathfrak{h}_\alpha{}^\rho\mathfrak{h}_\beta{}^\sigma\nabla_\rho\mathfrak{n}_\sigma$ as well as its trace as $K = \mathfrak{h}^{\alpha\beta}K_{\alpha\beta}$. Finally we can write

$$-K = \frac{(\chi^2-1)f_{,\chi\chi}}{a\,[1+f_{,\chi}^2\,(\chi^2-1)]^{\frac{3}{2}}} - \frac{3a_{,f}}{\sqrt{1+f_{,\chi}^2\,(\chi^2-1)}\,a^2} + \frac{2\left((\chi^2-1)^2 f_{,\chi}^2 + \frac{3\chi^2}{2} - 1\right) f_{,\chi}}{[1+f_{,\chi}^2\,(\chi^2-1)]^{\frac{3}{2}}\,\chi\,a} = -K_\infty\,. \tag{5.3}$$

We will solve this differential equation numerically later on, provided we also give the function $a = a(f)$ to be determined in the following. As for now, we will only assume there is such a solution. Once more, we need this equation to hold as we want to embed this GR solution in VCDM, and in this case, we need to require that $K = K_\infty$.

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We can now proceed by realizing that the hypersurface where the junction conditions take place can be described from the inside-metric point of view as happening for

$$\Phi(x^\mu) = \chi - \chi_s = 0\,, \tag{5.4}$$

where a constant χ_s denotes the star surface, so that the unit vector normal to this hypersurface can be written as

$$n_\alpha dx^\alpha = \frac{a}{\sqrt{1-\chi^2}}\,d\chi\,. \tag{5.5}$$

Out of this covector one can define the projector $h_\alpha{}^\beta = \delta_\alpha{}^\beta - n_\alpha n^\beta$, and the extrinsic curvature $\mathcal{K}_{\mu\nu}$. Also for the inside metric we can define three vectors

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a}\,, \qquad \text{where} \qquad y^a \in \{t, z, \theta_2\}\,, \tag{5.6}$$

or

$$e^\mu_t \partial_\mu = \partial_t\,, \qquad e^\mu_z \partial_\mu = \partial_z\,, \qquad e^\mu_{\theta_2} \partial_\mu = \partial_{\theta_2}\,. \tag{5.7}$$

By doing this, out of the twelve scalars

$$h_{ab} = h_{ba} = h_{\mu\nu} e^\mu{}_a e^\nu{}_b|_{\chi=\chi_s}\,, \qquad \mathcal{K}_{ab} = \mathcal{K}_{ba} = \mathcal{K}_{\mu\nu} e^\mu{}_a e^\nu{}_b|_{\chi=\chi_s}\,, \tag{5.8}$$

we can prove that for the inside metric the only nonzero terms are the following ones

$$h_{tt} = -a(f(t,\chi_s))^2 f_{,t}(t,\chi_s)^2\,, \tag{5.9}$$

$$h_{zz} = \frac{a(f(t,\chi_s))^2 \chi_s^2}{1-z^2}\,, \tag{5.10}$$

$$h_{\theta_2\theta_2} = a(f(t,\chi_s))^2 \chi_s^2\,(1-z^2)\,, \tag{5.11}$$

$$\mathcal{K}_{zz} = \frac{a(f(t,\chi_s))\chi_s\sqrt{1-\chi_s^2}}{1-z^2}\,, \tag{5.12}$$

$$\mathcal{K}_{\theta_2\theta_2} = a(f(t,\chi_s))\chi_s\sqrt{1-\chi_s^2}(1-z^2)\,. \tag{5.13}$$

It should be noticed that the component $\mathcal{K}_{tt}$ vanishes for the inside metric.

6 Matching conditions

6.1 Israel junction conditions

We now impose the standard Israel junction conditions at the surface of the star. This standard treatment in GR is justified also in VCDM since (i) we have set the trace of the extrinsic curvature of the constant-t hypersurface to the same constant value K_∞ everywhere in both sides of the surface of the star; and (ii) we shall in the next subsection require the continuity of the unit normal to the constant-t hypersurface across the surface of the star as (6.23)–(6.24) below.

For the outside metric let us first consider the following two elements

$$h^+_{zz} = \frac{R(t)^2}{1-z^2}\,, \tag{6.1}$$

$$h^+_{\theta_2\theta_2} = R(t)^2\,(1-z^2)\,. \tag{6.2}$$

If we match them with the inside metric we find

$$R(t) = \chi_s\, a(f(t,\chi_s))\,. \tag{6.3}$$

Let us now consider the expression

$$\mathcal{K}^+_{zz}\Big|_{r=R(t)} = \mathcal{K}^-_{zz}\big|_{\chi=\chi_s, a=R/\chi_s}\,. \tag{6.4}$$

Then one finds a quadratic equation for $\dot{T}$ as

$$\dot{T}^2 + \mathcal{A}(t)\,\dot{T} + \mathcal{B}(t) = 0\,. \tag{6.5}$$

By taking its time derivative, we can also find an equation for $\ddot{T}$. Eq. (6.5) can be solved algebraically for $\dot{T}$ as

$$\dot{T} = -\frac{\sqrt{1-\chi_s^2}\,\sqrt{1-\chi_s^2+\bar{\beta}^2-\bar{F}}\,\sqrt{\bar{F}} - (1-\chi_s^2+\bar{\beta}^2-\bar{F})\bar{\beta}}{(\bar{F}-\bar{\beta}^2)(1-\chi_s^2+\bar{\beta}^2-\bar{F})\,\bar{g}_2}\,\dot{R} + \frac{\bar{g}_1\dot{\kappa}}{\bar{g}_2}\,, \tag{6.6}$$

where a bar indicates that the function is evaluated at $r = R(t)$, e.g. $\bar{F} = F(t, r = R(t))$. We have picked up the solution which for vanishing β, leads to $\dot{T} > 0$ for $\dot{R} < 0$. Here and in the following we always assume that $\kappa(t) > 0$, so that $\bar{\beta} > 0$. On considering the possible value for $R(t) = R_h$ for which $\sqrt{\bar{F}} = \bar{\beta} > 0$, then at this instant of time the solution would be crossing the horizon. However, the quantity $\dot{T}/\dot{R}$ would still remain finite as

$$\dot{T} - \frac{\bar{g}_1\dot{\kappa}}{\bar{g}_2} = -\frac{(1-\chi_s^2+\bar{\beta}^2)}{(1-\chi_s^2)\left[\sqrt{\bar{F}}\sqrt{1+\frac{\bar{\beta}^2-\bar{F}}{1-\chi_s^2}} + \left(1+\frac{\bar{\beta}^2-\bar{F}}{1-\chi_s^2}\right)\bar{\beta}\right]}\,\frac{\dot{R}}{\bar{g}_2}\,. \tag{6.7}$$

On using the obtained relations for $\dot{T}$ and $\ddot{T}$, we also find that

$$\mathcal{K}^+_{tt} = 0\,, \tag{6.8}$$

satisfying automatically the matching conditions.

At this level all the components of the extrinsic curvature tensor are equal for both the inside/outside metrics. However, we still need to match

$$\mathcal{E}_{tt} \equiv h^+_{tt} - h^-_{tt}\Big|_{a=R/\chi_s} = 0\,. \tag{6.9}$$

Since $R(t) = \chi_s\, a[f(t,\chi_s)]$, we can replace

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{da}\,\frac{da}{df}\,f_{,t} = \chi_s\, a_{,f}\, f_{,t}\,. \tag{6.10}$$

This relation, together with the one obtained before for $\dot{T}$ gives the following constraint

$$\mathcal{E}_{tt} = \frac{a f_{,t}^2\left(a^4 K_\infty^2\chi_s^3 - 9a^2\chi_s^3 - 9a_{,f}^2\chi_s^3 + 9ar_H\right)}{a^3K_\infty^2\chi_s^3 - 9a\chi_s^3 + 9r_H} = 0\,. \tag{6.11}$$

This constraint can be solved as

$$\frac{3a_{,f}^2}{a^4} + \frac{3}{a^2} = \frac{K_\infty^2}{3} + \frac{3r_H}{a^3\chi_s^3}\,. \tag{6.12}$$

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This equation is the equation which determines a as a function of f, and it can be recognized as being the Friedmann equation of a closed universe with a cosmological constant and a dust component which can be written as

$$\frac{\rho_{\rm dust}}{M_{\rm P}^2} = \frac{3r_H}{a^3\chi_s^3} = \frac{3(2GM)}{R^3}\,, \qquad \text{or} \qquad M = \frac{4}{3}\pi\rho_{\rm dust}R^3\,. \tag{6.13}$$

If we define $a_{\rm max}$ as the maximum value of a, in this case $a_{,f}(a = a_{\rm max}) = 0$, which also implies that $\dot{R} = 0$. In this case, on the maximum, we also find

$$\frac{3r_H}{R_{\rm max}^3} = \frac{3r_H}{a_{\rm max}^3\chi_s^3} = \frac{3}{a_{\rm max}^2} - \frac{K_\infty^2}{3} = \frac{3\chi_s^2}{R_{\rm max}^2} - \frac{K_\infty^2}{3}\,, \tag{6.14}$$

which sets the value of χ_s as

$$\chi_s^2 = \frac{r_H}{R_{\rm max}} + \frac{K_\infty^2 R_{\rm max}^2}{9}\,. \tag{6.15}$$

6.2 Absence of cusp of $\mathcal{T}$-constant surface

As we can see from the VCDM action (2.6) introduced in section 2, in VCDM we need to add an extra junction condition. In fact, in VCDM there is a field $\mathcal{T}$ which is required to be timelike everywhere (and can be always fixed to be $\mathcal{T} = t$). Hence $\mathcal{T}$ has a non-trivial profile and we need this field to be continuous at the surface of the star. Out of $\mathcal{T}$, we define

$$\mathcal{A} = \frac{1}{\sqrt{-\partial_\rho\mathcal{T} g^{\rho\sigma}\partial_\sigma\mathcal{T}}}\,, \tag{6.16}$$

and the unit vector normal to constant-$\mathcal{T}$ hypersurfaces as

$$\mathfrak{n}_\mu = -\mathcal{A}\,\partial_\mu\mathcal{T}\,, \tag{6.17}$$

which on shell satisfies the following equation of motion

$$\nabla_\mu\mathfrak{n}^\mu = -\frac{3}{2}\lambda - \phi\,, \tag{6.18}$$

where λ and ϕ being the other two auxiliary field of VCDM. For a solution of VCDM which is also a solution of GR both λ and ϕ need to be constant. Then, let us consider both inside/outside the solution with λ and ϕ being constants everywhere. Then, on choosing Gaussian normal coordinates about the matching hypersurface we find

$$\partial_l\left[\sqrt{|h|}\mathcal{A}\partial^l\mathcal{T}\right] + \partial_s\left[\sqrt{|h|}\mathcal{A}h^{sr}\partial_r\mathcal{T}\right] = \left(\frac{3}{2}\lambda + \phi\right)\sqrt{|h|}\,, \tag{6.19}$$

where l is a coordinate orthogonal to the hypersurface. On integrating about a thin layer in l and since there are no singular terms we find

$$\sqrt{|h|}\mathcal{A}\partial^l\mathcal{T}\Big|_+ = \sqrt{|h|}\mathcal{A}\partial^l\mathcal{T}\Big|_-\,, \tag{6.20}$$

which leads to

$$\mathcal{A}\partial^l\mathcal{T}\Big|_+ = \mathcal{A}\partial^l\mathcal{T}\Big|_-\,, \tag{6.21}$$

or, if $\mathcal{T} = t$, the previous relation can be rewritten as

$$[\mathcal{A} n^\mu \partial_\mu t] = \left[\frac{n^\mu \partial_\mu t}{\sqrt{-h^{00} - (n^\sigma \partial_\sigma t)^2}}\right] = 0\,, \qquad \text{or} \qquad [n^\mu \partial_\mu t] = 0\,, \tag{6.22}$$

having used the continuity of the induced metric $h_{\mu\nu}$. This analysis shows that we need to impose one extra non-trivial matching condition. In VCDM, we choose $\mathcal{T} = t$ (both inside and outside) so that we have

$$[n^\mu \mathfrak{n}_\mu] = -\mathcal{A}[n^\mu \partial_\mu t] = 0\,, \tag{6.23}$$

$$[e^\mu{}_a h_\mu{}^\nu \mathfrak{n}_\nu] = -\mathcal{A}[e^\mu{}_a \partial_\mu t] = 0\,. \tag{6.24}$$

In summary, the normal to the $\mathcal{T}$-constant hypersurface is the same inside/outside the hypersurface. These conditions basically ensure the absence of cusp of the $\mathcal{T}$-constant hypersurface at the intersection with the surface of the star.

7 Analytical and numerical insight on the collapse

In what follows, we will neglect the effect of the cosmological constant, because its value only gives tiny corrections to the numerical results and will not change the qualitative picture. This leads to considering $K_\infty = 0$. Before focusing into the numerics we want to explicitly write down the condition

$$[n^\mu \partial_\mu t] = 0\,, \tag{7.1}$$

in our case. Let us also redefine for later convenience

$$\kappa(t) = \kappa_+ \kappa_1(t)\,, \tag{7.2}$$

where we remind the reader that (neglecting the effective cosmological constant)

$$F = 1 - \frac{r_H}{r} + \frac{\kappa(t)^2}{r^4}\,, \qquad \kappa_+ \equiv \frac{3}{16}\sqrt{3} r_H^2\,, \tag{7.3}$$

so that $\kappa_1(t)$ becomes dimensionless. Then this last constraint $[n^\mu \partial_\mu t] = 0$ can be used as to constrain the dynamics of $\kappa_1(t)$. The acceptable solutions can be written as

$$\kappa_1 = \pm \frac{4\sqrt{3}\{\chi_s[1 - \cos f(t,\chi_s)] + f_{,\chi}(t,\chi_s)\,(\chi_s^2 - 1)\sin f(t,\chi_s)\}[1 + \cos f(t,\chi_s)]^{\frac{3}{2}}}{9\chi_s^4 \sqrt{1 - \cos f(t,\chi_s)}\,\sqrt{1 + (\chi_s^2 - 1) f_{,\chi}(t,\chi_s)^2}}\,. \tag{7.4}$$

This shows that once the solution $f(t,\chi)$ is known, i.e., if we obtain an appropriate slicing which satisfies all necessary conditions, the behavior of κ_1 is also known.

For the internal solution, in the absence of a cosmological constant, we have already found in eq. (6.12) that

$$\frac{a_{,f}^2}{a^4} + \frac{1}{a^2} = \frac{r_H}{a^3 \chi_s^3}\,, \tag{7.5}$$

which is solved by

$$a = \frac{1}{2} a_{\max}\,[1 + \cos f]\,, \qquad 0 \le f < \pi\,, \qquad \text{and} \qquad a_{\max} = \frac{r_H}{\chi_s^3}\,, \tag{7.6}$$

having fixed the initial condition such that $a(f=0)=a_{\max}$. The singularity of $a=0$ is reached whenever $f=\pi$. Using this solution, we want to discuss, given a value of χ_s, the values of f for which the solution hits the apparent horizon. In fact, the apparent horizon is located at the solution for $R(t)=r_H$. In other words, we want to find the value of $f=f_H$ at which this happens. Then we need to solve

$$\frac{r_H}{\chi_s}=a(f=f_H)\,, \tag{7.7}$$

which then gives the following result

$$f=f_H\equiv\arccos(2\chi_s^2-1)\,. \tag{7.8}$$

Thus, for a given value of χ_s, the part of the interior solution with $f(t,\chi_s)>f_H(\chi_s)$ is inside the apparent horizon, as seen from the outer space.

Next, we substitute the function $a(f)$ inside eq. (5.3) as to find an elliptic equation f which can be rewritten as

$$f_{,\chi\chi}=\frac{2(1-\chi^2)\,f_{,\chi}^3}{\chi}-\frac{3f_{,\chi}^2\sin f}{1+\cos f}+\frac{(2-3\chi^2)f_{,\chi}}{\chi\,(\chi^2-1)}-\frac{3\sin f}{(\chi^2-1)\,(1+\cos f)}\,. \tag{7.9}$$

This equation does not involve any time derivative and thus can be solved on each constant-t hypersurface by imposing the following boundary conditions

$$f(t,\chi=0)=f_0(t)\,,\qquad f_{,\chi}(t,\chi=0)=0\,, \tag{7.10}$$

where here the suffix 0 stands for the center of the dust cloud. The second condition is imposed as to make the slicing non singular at the center of the star.

In case the collapse runs with $\chi\le\chi_s\ll1$, i.e. for $r_H\ll R_{\max}$, the Taylor series in χ provides an excellent approximation for the solution of f as

$$f=f_0(t)+\frac{1}{2}\left(\frac{\sin f_0(t)}{1+\cos f_0(t)}\right)\chi^2+\cdots+\mathcal{O}[\chi^{14}]\,, \tag{7.11}$$

where we have set boundary conditions as to have $\lim_{\chi\to0}f_{,\chi}=0$. We can also see that $\lim_{f_0\to0}f=0$. In the remaining section, in order to have a firm analytical insight, we will use this approximate solution.[2] On specifying the function $f_0(t)\equiv f(t,0)$, we will also solve eq. (7.9) numerically (for values of χ_s closer to unity). In any case the dependence of f on time is only through the function $f_0(t)$.

Then the function $f(t,\chi)$, both analytically and numerically, does depend on t only through $f_0(t)$, therefore we have

$$f(t,\chi)=\tilde{f}(f_0(t),\chi)\,. \tag{7.12}$$

This is because, as already mentioned, (7.9) with (7.10) can be integrated on each constant-t hypersurface. Hereafter, for simplicity of notation we express the function on the right hand side of (7.12) as $f((f_0(t),\chi))$ by omitting the tilde.

In the remaining part we will only consider the positive solution for κ_1 defined inside eq. (7.4), so that we have

$$\kappa_1=\tilde{\kappa}_1(f_0(t),\chi_s)\,. \tag{7.13}$$

[2]Numerically, the Taylor series in this solution converges slowly because of the negative powers of $1+\cos f_0$. This solution nonetheless gives the same qualitative results as the numerical results, for values of $0<\chi_s\lesssim10^{-2}$.

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Hereafter, for simplicity we express the function on the right hand side as $\kappa_1((f_0(t),\chi))$ by omitting the tilde. In this section, when we do numerics, we will set $f_0(t) = t$, as to try to reach any point of the solution (making omitting the tilde justified). We know that in any case $0 \leq f < \pi$. However, in the analytic description that follows, we will study the behavior of the solution without specifying any explicit dependence of f_0 on time.

7.1 Vanishing lapse as final point of the collapse

Let us consider once more the lapse for the exterior solution (for any value of r). Then one finds

$$N(t,r) = \alpha(t,r)\,\dot{T}(t)\,, \tag{7.14}$$

where

$$\alpha = \left(1 - \frac{\dot{\kappa}}{\dot{T}}\int_\infty^r \frac{1}{r_1^2} F(t,r_1)^{-3/2}\mathrm{d}r_1\right)\sqrt{F}\,, \tag{7.15}$$

$$F = 1 - \frac{r_H}{r} + \frac{\kappa(t)^2}{r^4}\,, \tag{7.16}$$

so that $\lim_{r\to\infty}\alpha = 1$. We also set boundary conditions for $N(t,r)$ so that $\lim_{r\to\infty} N(t,r) = 1$. This means that t is the cosmological time, that is the time as measured by an observer far away from the black hole. In this case we have that

$$\dot{T} = 1\,. \tag{7.17}$$

This, combined with (6.6), leads to

$$1 = -\frac{[\kappa^2 R_{\rm max} + R^4(R_{\rm max} - r_H)]RR_{,f_0}\dot{f}_0}{\sqrt{R^4 - R^3 r_H + \kappa^2}\left[\sqrt{R^4 - R^3 r_H + \kappa^2}\,\sqrt{R}\,\sqrt{R_{\rm max} - r_H}\,\sqrt{r_H(R_{\rm max} - R)} + r_H\kappa\,(R_{\rm max} - R)\right]} + \left(\int_\infty^R \frac{1}{r_1^2}F(t,r_1)^{-3/2}dr_1\right)\kappa_{,f_0}\dot{f}_0\,. \tag{7.18}$$

On doing this, $f_0(t)$ depends on the collapse parameters: indeed this last condition corresponds to an ODE for f_0, which can be written as

$$\dot{f}_0 = -\frac{1}{\left(\displaystyle\int_0^{1/R} F\left(t,\tfrac{1}{u}\right)^{-3/2}du\right)\kappa_{,f_0} + \dfrac{[\kappa^2 R_{\rm max} + R^4(R_{\rm max} - r_H)]RR_{,f_0}}{\sqrt{R^4 - R^3 r_H + \kappa^2}\left[\sqrt{R^4 - R^3 r_H + \kappa^2}\,\sqrt{R}\,\sqrt{R_{\rm max} - r_H}\,\sqrt{r_H(R_{\rm max} - R)} + r_H\kappa\,(R_{\rm max} - R)\right]}}\,, \tag{7.19}$$

where we have changed the integration variable from r_1 to $u = 1/r_1$. We can insert this value of $\dot{f}_0$ into the expression in eq. (7.15) for the lapse N (or α) through $\dot{\kappa} = \dot{f}_0\kappa_{,f_0}$, where on

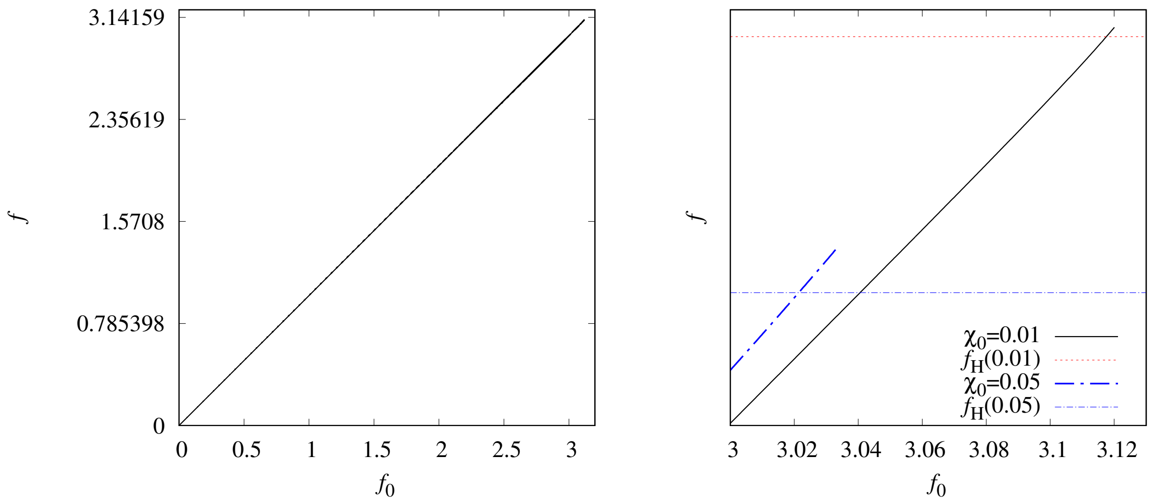

Figure 1. Plot of the function $f_s = f(f_0(t), \chi_s)$. In the left panel, we show that in the allowed range of f_0, f_s and f_{s,f_0} are always finite for a fixed value of $\chi_s = 0.01$. This behavior holds for other values of χ_s as well. In the right figure, we show that the collapse ends inside the apparent horizon of the metric, this because $f > f_H$. Once more this behavior hold for all values of χ_s such that $0 < \chi_s < 1$.

the right hand side we have considered κ as a function of f_0 and χ, finding

$$N = \left[1 - \frac{\kappa_{,f_0} \displaystyle\int_0^{1/r} F\left(t, \frac{1}{u}\right)^{-3/2} du}{\kappa_{,f_0} \displaystyle\int_0^{1/R} F\left(t, \frac{1}{u}\right)^{-3/2} du + \frac{\left[\frac{\kappa^2}{R^4} R_{\max} + (R_{\max} - r_H)\right] R R_{,f_0}}{\sqrt{F(t,R)}\left[\sqrt{F(t,R)}\,\sqrt{R}\,\sqrt{R_{\max}-r_H}\,\sqrt{r_H(R_{\max}-R) + r_H \frac{\kappa}{R^2}(R_{\max}-R)}\right]}} \right] \times \sqrt{F(\kappa, r)} \,. \tag{7.20}$$

The function $f_{,f_0}$ both analytically and numerically keeps finite during the whole evolution, as it is shown in figure 1.

Therefore for $\kappa_1^2 > 1$, that is for $\kappa > \kappa_+$ (assuming positive κ_1), the integral is always well behaved. If $0 < \kappa_1 < 1$ the integral and the squared roots are well behaved as long as $R(t)$ does not coincide with one of the two real roots of $F(t, r) = 1 - r_H/r + \kappa(t)^2/r^4$. In fact, also the other term in the denominator, the one proportional to $R_{,f_0}$ tends to blow up once more if $R(t) \neq 0$ (or $f < \pi$) and $R(t)$ is such that $F(t, r = R(t)) = 0$. Let us then consider the dynamics of κ_1, where

$$\kappa = \kappa_+ \kappa_1\big(f(f_0, \chi_s)\big) \,, \qquad \text{and} \qquad \kappa_+ = \frac{3\sqrt{3}}{16} r_H^2 \,. \tag{7.21}$$

We find that it depends on the value of χ_s. So let us now study this behavior in some detail.

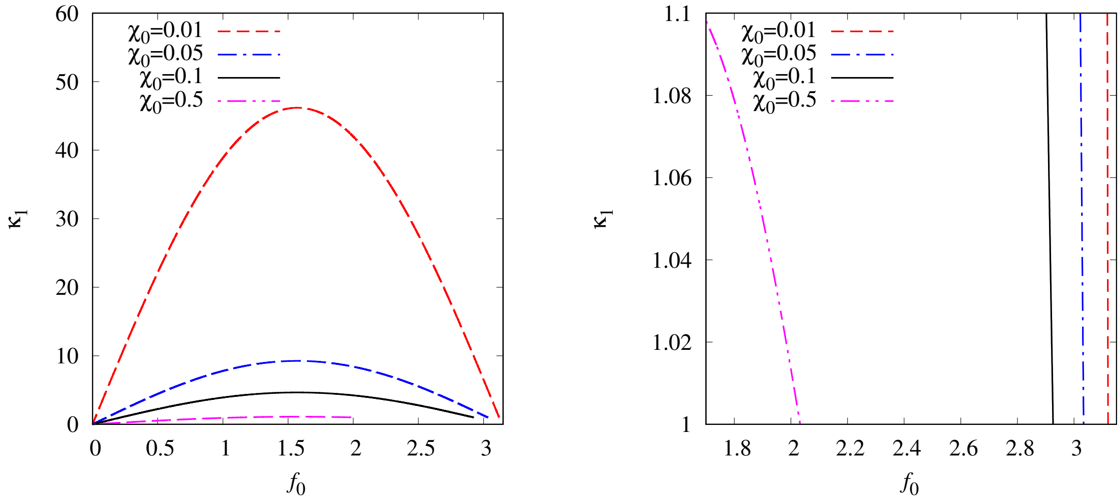

Figure 2. Plots of κ_1 as a function of f_0 for different values of χ_s. In the left panel we can see that κ_1 starts from zero, grows up to a maximum (which is larger than unity) and then κ_1 decreases up to the point it reaches the value of unity (as it can be seen in the right figure). This final value of κ_1 being unity takes place when the solution has already entered its own apparent horizon, as discussed previously in figure 1. In other words, at horizon crossing, we find that $\kappa_1(t) > 1$.

7.1.1 Small values of χ_s

For some values of χ_s, with $\chi_s \lesssim 0.65$, we can distinguish the following stages, which can be seen in figure 2.

1. First stage: initial collapse. At this stage $\kappa_1(f_0 = 0) = 0$ and $\kappa_{1,f} > 0$. This stage does not show any problem as $R \approx R_{\rm max}$ and $F > 0$. Soon κ_1 reaches values larger than unity (for which F cannot vanish any more), but κ_{1,f_0} tends to reduce up to a time (or a value of f_0) where it flips sign.

2. Second stage: $\kappa_{1,f_0} < 0$ up to horizon crossing. After κ_1 has reached a maximum (larger than unity) indeed κ_1 starts decreasing while remaining larger than unity. In this case F never vanishes. At horizon crossing, that is when $f = f_H$, still $\kappa_1 > 1$, so the collapsed star enters its own horizon.

3. Third stage: soon after the solution enters its own horizon, then $\kappa_1(f_0)$ keeps decreasing up to the value of $f_0 = f_{0F}$ at which $\kappa_1(f_0 = f_{0F}) = 1$. In this case, F vanishes for the value of r_{0F}, $r_{0F} = 3r_H/4$, such that $R(f_{0F}) < r_{0F} < r_H$. Indeed one can study both R and κ_1 as functions of f_0. One sees that $\kappa_1(f_0)$ reduces but $F(f_0, r = R(f_0))$ reaches its minimum when $\kappa_1(f_0)$ is still larger than unity. Then $R(f_0)$ will be such that $F(f_0, r = R(f_0))$ will be located to the right of the minimum of the function $F(f_0, r)$. However, as already stated above, f_0 reaches a point such that $\kappa_1(f_{0F}) = 1$. At this time there exists an $r = r_{0F}$ at which $F(f_0, r = r_{0F}) = 0$. This point makes the integral in the denominator of $\dot{f}_0$ blow up and also makes $\dot{f}_0$ vanish. When this happens for some value of $f_0 = f_{0F}$, one finds that $\kappa_1 = 1$, however the surface of the star (represented as a dot in figure 3) is located at $R(f_{0F})/r_H < 3/4$.

In figure 3 we plot the function $F(t, r)$ (which depends on time via the function $\kappa(t)$) at the instant of time when κ_1 becomes equal to unity as to make the integral in eq. (7.19) diverge.

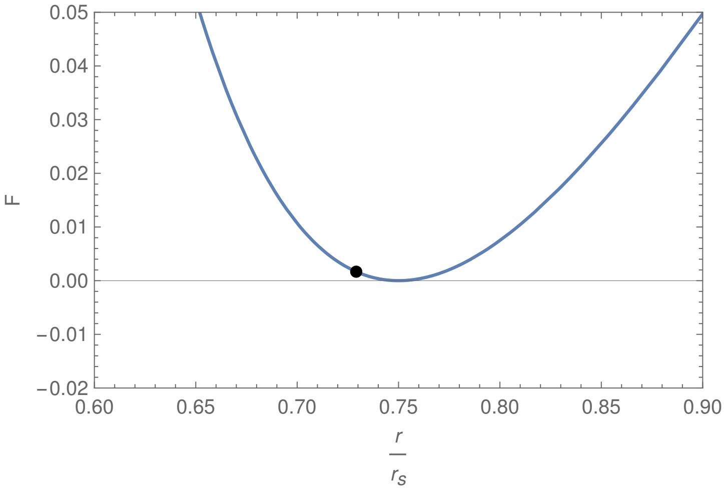

Figure 3. Reaching the point at which $\dot{f}_0$ vanishes, for small values of χ_s. In fact, $\dot{f}_0 \to 0$ at a value of $f_0 = f_{0F}$ such that $\kappa_1(f(f_{0F}, \chi_s)) = 1$. In this case the integral $\int_0^{1/R(t)} F\left(t, \frac{1}{u}\right)^{-3/2} du$ diverges as the variable u crosses the value $4/(3r_H)$, since $\kappa_+ \to 1$, and $R(f_{0F}) < 3r_H/4$ as shown as a dot in the plot.

As a consequence, as we will discuss below, as $\kappa_1 \to 1$ the lapse tends to vanish, i.e. $N \to 0$, and for the cosmological observer the singularities (e.g. when a vanishes) are never reached, as the solutions will take an infinite time to reach f_{0F}, i.e. the left zero-F point, as $\dot{f}_0 \to 0$.

7.1.2 Large values of χ_s

For larger values of χ_s, i.e. $0.65 \lesssim \chi_s < 1$, the dynamics of $\kappa_1(f_0)$ does change and, in this case, the collapse works in a different way. In fact, we have that $\kappa_1(f_0)$ remains always smaller than unity. This seems to be giving problems since the beginning of the collapse. However, the collapse starts at $R(f_0 = 0) = R_{\rm max} > r_H$ so that at least initially $R_{\rm max}$ is not one of the two roots of F. In other words, if $\kappa_1 < 1$, F can vanish but it does not as long as R does not reach a minimum critical value, $r_{1,+} = R(f_{0,+})$, at which $F = 0$. We know that if $0 < \kappa_1 < 1$, the function F vanishes at two values of r, $r_{F\pm}$ so that $r_{F-} < \frac{3}{4} r_H < r_{F+}$ and $r_{F+} - r_{F-} \to 0$ as $\kappa_1 \to 1^-$. The collapse stops when, at a given value of $f_0 = f_{0,+}$, we have that $\kappa_1(f_{0,+}) < 1$ and $F(\kappa_1(f_{0,+}), R(f_{0,+})) = 0$. In fact, we do have that, because of the collapse, $R_{,f_0} < 0$ and $R(f_0)$ reaches the larger root of F, $R = R(f_{0,+})$, before κ_1 reaches unity, and this happens indeed at $f_0 = f_{0,+}$. At this point indeed once more both terms in the denominator of $\dot{f}_0$ in eq. (7.19) tend to diverge as R approaches the largest root of $F(\kappa_1(f_{0,+}), R(f_{0,+})) = 0$. Figure 4 is devoted to this case.

7.2 Limiting surface

Here we find the limiting surface, i.e. the surface with the largest radius at which N vanishes. We consider both the small and large values for χ_s.

7.2.1 Small values of χ_s

Let us consider here the case of small χ_s, and define f_{0F}, that is the value of f_0 at which $\kappa_1(f_{0F}) = 1$. Then let us also define $R_{0F} = R(f_0 = f_{0F})$. In general the lapse N (or α) is a function of time (or f_0) and r (as seen from the outer metric point of view). Hence, let

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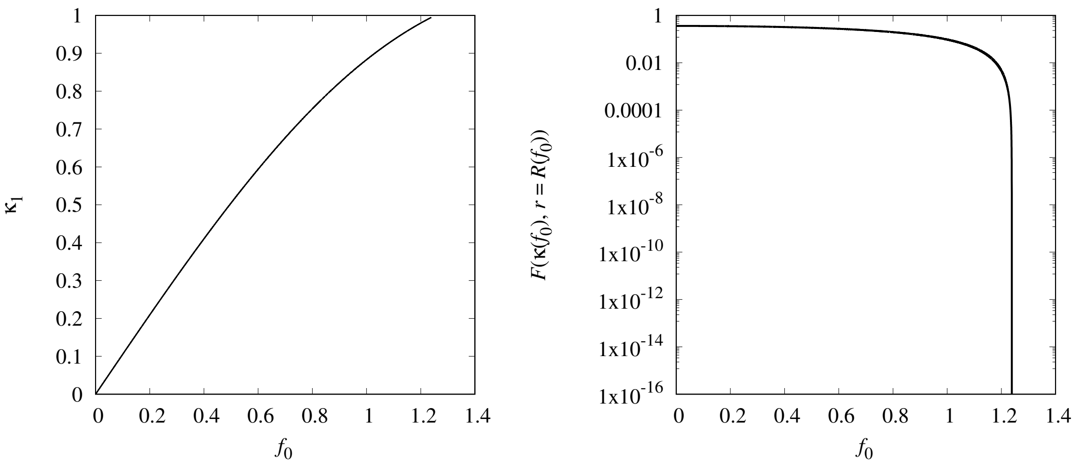

Figure 4. In the left panel, we show a plot of the function $\kappa_1(f_0)$ for a value of $\chi_s = 0.8$. It is clear that although κ_1 increases, in the allowed domain of f_0 it never reaches unity. In the right panel, we instead show, for the same value of χ_s, the behavior of $F(f_0)$, and it is clear that it vanishes while $0 < \kappa_1 < 1$. At this point, that we name as $f_0 = f_{0,+}$, which still happens when the solution has already entered its apparent horizon, we have that $\lim_{f_0 \to f_{0,+}} F(\kappa_1(f_0), R(f_0)) = 0$, and as such we have that $\lim_{f_0 \to f_{0,+}} \dot{f}_0 = 0$.

us evaluate the lapse at $f_0 = f_{0F}$ but for $r = r_+ > R_{0F}$ at which $F(f_{0F}, r_+) = 0$, that is $N(f_{0F}, r_+)$ and $r_+ = \frac{3}{4} r_H$. Notice that in this case $F(f_{0F}, r = R_{0F}) \neq 0$ in general. Then setting $F_+(t,r) = F(t, r : \kappa = \kappa_+)$, we have

$$
\begin{aligned}
N(f_{0F}, r_+) &= \left(1 - \kappa_{,f_0} \dot{f}_0 \int_\infty^{r_+} \frac{1}{r_1^2} F(t, r_1)^{-3/2} \mathrm{d}r_1\right) \sqrt{F(f_{0F}, r_+)} = \sqrt{F(f_{0F}, r_+)} \\
&\times \left(1 - \frac{\kappa_{,f_0} \displaystyle\int_0^{1/r_+} F_+\left(t, \frac{1}{u}\right)^{-3/2} du}{\kappa_{,f_0} \displaystyle\int_0^{1/R} F_+\left(t, \frac{1}{u}\right)^{-3/2} du + \frac{\left[\frac{\kappa_+^2}{R^4} + (R_{\max} - r_H)\right] R R_{,f_0}}{\sqrt{F_+(t,R)}\left[\sqrt{F_+(t,R)}\sqrt{R}\sqrt{R_{\max} - r_H}\sqrt{r_H (R_{\max} - R)} + r_H \frac{\kappa_+}{R^2} (R_{\max} - R)\right]}}\right) \\
&\approx \left(1 - \frac{\displaystyle\int_0^{1/r_+} F_+\left(t, \frac{1}{u}\right)^{-3/2} du}{\left(\displaystyle\int_0^{1/r_+} F_+\left(t, \frac{1}{u}\right)^{-3/2} du + \int_{1/r_+}^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du\right)}\right) \sqrt{F_+(f_{0F}, r_+)} \\
&= \left(1 - \frac{1}{1 + C_0}\right) \sqrt{F_+(f_{0F}, r_+)} = \frac{\sqrt{F_+(f_{0F}, r_+)}}{2} = 0\,,
\end{aligned}
\tag{7.22}
$$

because $F_+(r_+) = 0$, and

$$C_0 = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_{1/(r_+-\epsilon)}^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du}{\displaystyle\int_0^{1/(r_++\epsilon)} F_+\left(t, \frac{1}{u}\right)^{-3/2} du} = 1\,. \tag{7.23}$$

This shows that for small χ_s we have $N(f_{0F}, r_+) = 0$.

Let us now evaluate the lapse N at $r = R_{0F}$, that is $N(f_{0F}, R_{0F})$. As a difference from the previous calculation, we have that $F(f_0, R_{0F}) \neq 0$ but finite as seen also in figure 3. We have that

$$\begin{aligned} N(f_{0F}, R_{0F}) &= \left(1 - \kappa_{,f_0} \dot{f}_0 \int_\infty^{R_{0F}} \frac{1}{r_1^2} F_+(t, r_1)^{-3/2} \mathrm{d}r_1\right) \sqrt{F_+}\,, \\ &\approx \left(1 + \kappa_{,f_0} \frac{1}{\left(-\displaystyle\int_0^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du\right) \kappa_{,f_0}} \times \int_0^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du\right) \sqrt{F_+} \\ &= 0\,. \end{aligned} \tag{7.24}$$

This shows that $N(f_{0F}, R_{0F}) = 0$, with $R_{0F} < r_+$.

Finally, on evaluating the lapse N at an intermediate point $R_{0F} < r_{0,+} < r_+$, we find

$$\begin{aligned} N(f_{0F}, r_{0,+}) &= \left(1 - \kappa_{,f_0} \dot{f}_0 \int_\infty^{r_{0,+}} \frac{1}{r_1^2} F_+(t, r_1)^{-3/2} \mathrm{d}r_1\right) \sqrt{F_+}\,, \\ &\approx \sqrt{F_+} \left[1 - \frac{1}{\left(\displaystyle\int_0^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du\right)} \right. \\ &\qquad \left. \times \left(\int_0^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du + \int_{1/R_{0F}}^{1/r_{0,+}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du\right)\right] \\ &\approx (1-1) \sqrt{F(r_{0,+})} = 0\,, \end{aligned} \tag{7.25}$$

since

$$\frac{\displaystyle\int_{1/R_{0F}}^{1/r_{0,+}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du}{\displaystyle\int_0^{1/R_{0F}} F_+\left(t, \frac{1}{u}\right)^{-3/2} du} = 0\,, \tag{7.26}$$

as the numerator is finite. Then this means that, at $f_0 = f_{0F}$, $N \to 0$ from the origin up to $r_+ = \frac{3}{4} r_H$. So the limiting surface is $r = r_+$. This result then leads to the conclusion that the metric, at the end of the collapse (which takes an infinite time t) coincides with the spherical symmetric static solution having $\kappa = \kappa_+$.

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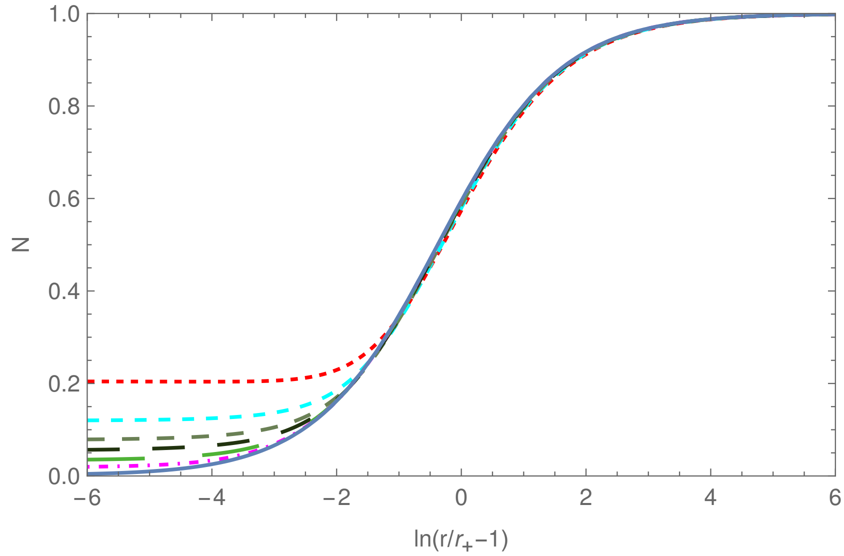

Figure 5. The lapse function N in the outer region ($r > r_+ = \frac{3}{4} r_H$) is plotted for several values of f_0 ($f_0 = 3.116$, 3.118, 3.119, 3.1195, 3.11986, 3.12001, 3.12006292) as $f_0 \to f_{0F}$.

In particular, on plotting the N as a function of r for the outer metric we can see that $N \to 0$ as $r \to r_+$ as shown in figure 5.

7.2.2 Large values of χ_s

For large values of χ_s let us evaluate the lapse at $f_0 = f_{0,+}$ where $F(f_{0,+}, R(f_{0,+})) = 0$. Here we use time $f_0(t)$ instead of t to evaluate the lapse function. At this point, setting $\kappa_{0,+} = \kappa(f_{0,+})$, the lapse $N(f_{0,+}, R_{0+})$ is given by

$$\begin{aligned} N(f_{0,+}, R_{0+}) &= \left(1 - \kappa_{,f_0} \dot{f}_0 \int_\infty^{R_{0+}} \frac{1}{r_1^2} F\left(f_{0,+}, r_1 : \kappa_{0,+}\right)\right)^{-3/2} \mathrm{d}r_1 \Bigg) \sqrt{F\left(f_{0,+}, R_{0+} : \kappa_{0,+}\right)} \\ &\approx \left(1 - \frac{\kappa_{,f_0} \displaystyle\int_0^{1/R_{0+}} F\left(f_{0,+}, \frac{1}{u} : \kappa_{0,+}\right)^{-3/2} du}{\kappa_{,f_0} \displaystyle\int_0^{1/R_{0+}} F\left(f_{0,+}, \frac{1}{u} : \kappa_{0,+}\right)^{-3/2} du + \frac{[\kappa^2 R_{\max} + R_{0+}^4 (R_{\max} - r_H)] R_{0+} R_{,f_0}}{\sqrt{R_{0+}^4 - R_{0+}^3 r_H + \kappa_{0,+}^2}\,[r_H \kappa_{0,+} (R_{\max} - R_{0+})]}} \right) \\ &\quad \times \sqrt{F\left(f_{0,+}, R_{0+} : \kappa_{0,+}\right)} = 0\,, \end{aligned} \tag{7.27}$$

because the quantity in parenthesis is of order unity, but $\sqrt{F} \to 0$. Along the same lines one can show that also for the internal metric N vanishes as $\dot{f}_0 \to 0$, so that the lapse, at $f_0 = f_{0,+}$, vanishes up to $r = R_{0+}$. At this time we find that $\kappa_1 < 1$. Therefore, for large χ_s's the end of collapse asymptotically (in time-t) tends to a static metric having $0 < \kappa_1 < 1$. For

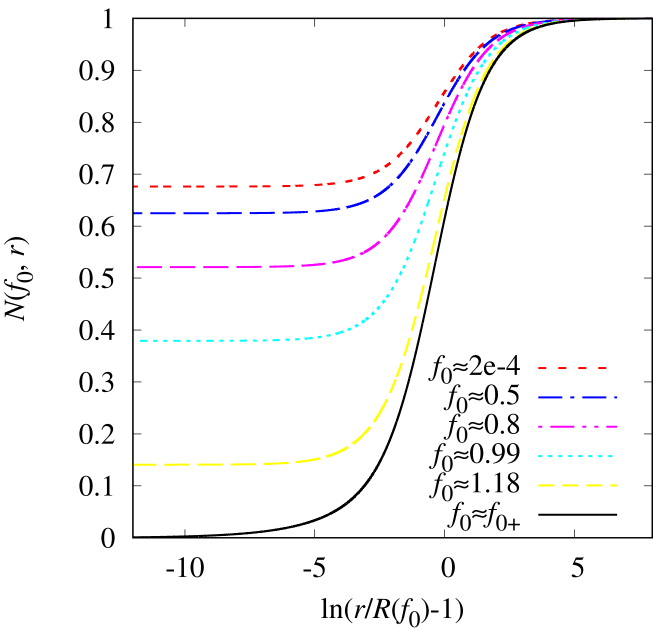

Figure 6. The lapse function N in the outer region for large χ_s ($\chi_s = 0.8$), for several values of f_0.

other values of r (and f_0), we need to evaluate

$$N(f_0,r) = \sqrt{F(\kappa(f_0),r)} \times \left(1 - \frac{\kappa_{,f_0}\int_0^{1/r} F\left(f_0,\frac{1}{u}\right)^{-3/2} du}{\kappa_{,f_0}\int_0^{1/R} F\left(f_0,\frac{1}{u}\right)^{-3/2} du + \frac{\left[\frac{\kappa^2}{R^4}R_{\max}+(R_{\max}-r_H)\right]RR_{,f_0}}{\sqrt{F(f_0,R)}\left[\sqrt{F(f_0,R)}\,\sqrt{R}\sqrt{R_{\max}-r_H}\,\sqrt{r_H(R_{\max}-R)+r_H\frac{\kappa}{R^2}}\,(R_{\max}-R)\right]}} \right), \tag{7.28}$$

where $R = R(f(f_0))$ stands for the radius at the surface of the star. We plot $N(f_{0+}, r)$ as a function of r, in figure 6.

8 Discussion and conclusions

The VCDM theory is a Type IIa minimally modified gravity (MMG). By type IIa we mean a theory for which an Einstein frame does not exist, i.e. we cannot rewrite the theory by means of the Einstein-Hilbert action and matter fields coupled to gravity in a whatever non-trivial way, and in which the propagation of gravitational waves is the same as the propagation of electromagnetic waves [7]. Furthermore it is a MMG theory, which means that in the gravity sector, the theory only has two local physical degrees of freedom, the two polarizations of the standard tensorial gravitational waves. In other words, this theory then does not add any additional degrees of freedom in the gravity sector (as happens instead in standard

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scalar-vector-tensor theories). The fact, the absence of extra degrees of freedom in the gravity sector, means we do not need a mechanisms as to screen them. On the other hand, one needs to see the solutions of VCDM which can be tested against observations. For instance, one needs to find VCDM solutions which describe black holes and stars. Such solutions, assuming spherical symmetry, are known to exist and they coincide with GR solutions if both the trace of the extrinsic curvature (for t-constant slicing) and the auxiliary field ϕ are constant [4, 6]. The property that VCDM and GR share common solutions was shown in [3], and it always holds in any spacetime geometry provided that the auxiliary fields of VCDM, ϕ and λ, are constant in time and space and that the time-t foliation of the manifold admits a constant trace for the extrinsic curvature, namely $K = K_\infty$. In this paper, we have shown that we can successfully construct VCDM solutions for a spherically symmetric collapse which are identical to the analogous solutions in GR. These solutions consist of a spherically symmetric collapse endowed with a foliation which keeps a constant extrinsic curvature during the collapse itself.

In particular, as in the Oppenheimer-Snyder case, we have a cloud of dust with initial radius $R_{\rm max}$. For the inside (the dust cloud) metric, we have rewritten a closed homogeneous and isotropic metric in a coordinate system which has a time-t slicing with $K = K_\infty$. For the outer solution we have instead rewritten the standard Schwarzschild-de Sitter metric once so that its time-t slicing allows $K = K_\infty$. Then we use Israel junction conditions to find the appropriate matching conditions at the surface of the star. In addition, for VCDM we have to make sure that all the fields are smooth at the matching surface, in particular that the constant-t hypersurface does not have any cusp at the junction surface.

In a previous work, [6], it was shown that the stationary spherically symmetric solutions in VCDM are Schwarzschild solutions written in the following particular coordinate system

$$ds^2 = -\frac{N^2}{F}\,[F-\beta^2]\,dt^2 + 2\,\frac{N\beta}{F}\,dt\,dr + \frac{dr^2}{F} + r^2\left[\frac{dz^2}{1-z^2} + (1-z^2)\,d\theta_2^2\right], \tag{8.1}$$

$$F = 1 - \frac{r_H}{r} + \frac{\kappa_0^2}{r^4}\,, \tag{8.2}$$

$$\beta = \frac{\kappa_0}{r^2}\,, \tag{8.3}$$

$$N = \sqrt{F}\,, \tag{8.4}$$

where we have neglected the contribution coming from the effective cosmological constant. We have a totally free real parameter κ_0. In this paper we have shown that on introducing $\kappa_1 = \kappa_0/\kappa_+$ with $\kappa_+ \equiv \frac{3}{16}\sqrt{3}r_H^2$, we see that for $t \to \infty$, the collapsing time-dependent solution approaches the static case solution with $0 < \kappa_1 < 1$ (and mirror case for negative κ_1's).

More in detail, from the point of view of the (far-away-from-the-star) cosmological observer, whose time corresponds to t, the surface of the star always enters its own apparent horizon (located at $r = r_H$) and keeps evolving until it reaches a configuration for which the lapse $N \to 0$. However, reaching this point takes an infinite time t. This point does not corresponds to the standard curvature singularity of the Oppenheimer-Snyder solution. We have already mentioned that this VCDM collapsing solution (i.e. not only the final static case but also the time-dependent collapsing one) is also present in GR; however, in GR, this behavior would corresponds merely to the artifact of the coordinate/foliation choice. Instead in VCDM, that breaks 4D-diffeomorphism, this foliation has physical meaning as the foliation which is chosen by the shadowy mode present in the theory. Different foliations correspond to intrinsically different objects in VCDM, and not all foliations of a given GR metric correspond to solutions of the VCDM equations of motion.

The presence of the vanishing lapse endpoint implies the necessity of a UV completion to describe the physics inside the black hole beyond this point. On the other hand, since the cosmic time t at the formation of this endpoint is infinite, VCDM can safely describe the whole history of the universe at large scales without knowledge of the unknown UV completion, despite the presence of the so-called shadowy mode whose description requires proper boundary conditions. The same final state could be a possible outcome for other theories which are endowed with shadowy modes, and further investigations in this sense could be interesting.

As stated above the final state of the solutions predicts that $0 < \kappa_1 < 1$. In particular, this parameter affects the exterior VCDM solutions, in particular it may influence the behavior of gravitational waves, and if so, it is actually possible to look for experimental bounds via the study of gravitational waves. We will defer such a study to a future project.

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A Sturm theorem

Let us study in a bit more details the function

$$F(t,r) = 1 - \frac{r_H}{r} + \frac{\kappa_+^2 \kappa_1(t)^2}{r^4}\,, \tag{A.1}$$

and in particular let us use the Sturm theorem as to determine the number of real solutions of the equation $r^4 F = 0$. In particular, let us study the zeros of $r^4 F$ in the case $r \neq 0$ and $\kappa_1 \neq 0$. Then we have

$$r^4 - r_H r^3 + \kappa_+^2 \kappa_1(t)^2 = 0\,. \tag{A.2}$$

The discriminant of this polynomial is given by

$$\Delta = \frac{3^9 \kappa_1^4 \left(\kappa_1^2 - 1\right) r_H^{12}}{2^{16}}\,. \tag{A.3}$$

Then for $0 < \kappa_1^2 < 1$ two solutions are real and two are complex. For $\kappa_1^2 = 1$, there are (at least) two coincident roots.

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Let us now study the case $\kappa_1^2 > 1$. According to Sturm theorem on univariate polynomials with real coefficients, one defines

$$P_0 = r^4 - r_H r^3 + \kappa_+^2 \kappa_1(t)^2 \,, \tag{A.4}$$

$$P_1 = P_0' = 4r^3 - 3r_H r^2 \,, \tag{A.5}$$

$$P_2 = -\mathrm{rem}(P_0, P_1) = \frac{3}{16} r^2 r_H^2 - \frac{27}{256}\kappa_1^2 r_H^4 \,, \tag{A.6}$$

$$P_3 = -\mathrm{rem}(P_1, P_2) = -\frac{9}{4}\kappa_1^2 r r_H^2 + \frac{27}{16}\kappa_1^2 r_H^3 \,, \tag{A.7}$$

$$P_4 = -\mathrm{rem}(P_2, P_3) = \frac{27}{256}\left(\kappa_1^2 - 1\right) r_H^4 \,. \tag{A.8}$$

Then the sign of these polynomials at $(-\infty, +\infty)$ is given by $S_1 = (+,-,+,+,+)$ and $S_2 = (+,+,+,-,+)$. So the number of real roots of P_0 is given by the difference between the number V of sign variations inside $S_{1,2}$, namely[3] $V(-\infty) - V(+\infty) = 2 - 2 = 0$. So for $\kappa_1^2 > 1$ we have no real roots.

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