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An extensive analysis of Schwarzschild exterior solution

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A R T I C L E   I N F O

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A B S T R A C T

We pursue a detailed analysis of the Schwarzschild geometry around a spherically symmetric, non-rotating, uncharged source and aim to construct the Schwarzschild metric by considering trigonometric, hyperbolic and logarithmic functions of position and time and solving the Einstein field equation. We investigate whether the Schwarzschild exterior solution is indeed independent of the choice of the nature of the function for the first two metric elements in the general expression for the Schwarzschild metric, and is solely dependent upon the centrally symmetric nature of the geometry taken into account.

1. Introduction

Einstein's 1915 paper on general theory of relativity revolutionized our understanding of our universe in an extraordinary way by providing a geometric and dynamic description of the fabric of spacetime. The absolute beauty of general relativity arises from the fact that gravity is nothing but a manifestation of the curvature of spacetime. The presence of matter causes spacetime to curve or bend around it and any second object approaching the former one experiences gravitational acceleration due to traversing the curved path along spacetime which can be geometrically visualized as falling towards the source along the curvature.

In the absence of any matter or energy, the spacetime is ideally flat, known as the 4D Minkowski spacetime. In this flat spacetime, when any matter/energy is introduced, curvature appears as an inevitable consequence, which is perceived as the gravitational field due to the source. As such, it can be rightly concluded that gravity is a manifestation of spacetime.

Now, the source of matter/energy can be considered across significant variations, such as spherically symmetric non-rotating static sources, dynamic rotating sources, charged rotating sources and so on. The nature of the source drastically influences the effects it has on the curvature of spacetime, as, for instance, rotating sources such as massive black holes drag spacetime around them. The impact of the source of matter on spacetime can be well understood from their solutions to Einstein's field equations.

For a spherically symmetric non-rotating source, the Einstein field equation has a solution, which is the metric function $g_{\mu\nu}$ for the geometry of spacetime outside the centrally symmetric spherical source, known as the Schwarzschild exterior solution to the Einstein field equation as elegantly derived by Karl Schwarzschild (1916). Such a source will generate a centrally symmetric gravitational field. Czerniawski (2006) in his work has discussed various approaches to deriving the Schwarzschild metric, in excellent detail.

Expression for the spacetime interval with metric signature $(+---)$ is given as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu}$ is metric tensor.

For the Schwarzschild exterior solution, it was considered that the matter distribution is centrally symmetric, static, non-rotating source. Since the spacetime is considered static, the components of the metric tensor should remain independent of time. As such the

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spacetime interval should remain invariant with the time reversal transformation $t \to -t$. So all cross terms involved in the definition of the interval goes to zero as $drdt \neq -drdt, dtd\theta \neq -dtd\theta, dtd\phi \neq -dtd\phi$, (Carroll, 2019). Thus, whether in the past or the future of the universe, the spacetime interval has to remain invariant, owing to the static, stationary assumption, which would imply that metric of the geometry must be independent of time.
Further, the space considered is homogeneous, the interval ds^2 should remain invariant under spatial translation. Which means whether we move forward or backward along the radial distance, the metric should remain an invariant quantity. The transformation $r \to -r$ to maintain the interval invariant, leading to the conclusion that the cross terms involving dr must vanish to preserve the homogeneity: (As $drd\theta \neq -drd\theta$).
On the hyper surface of constant r and t, which results in a two-sphere, the interval along the surface must remain invariant under rotational symmetry. To preserve rotational symmetry, the interval has to be independent of cross terms involving θ and ϕ (As $d\phi d\theta \neq -d\phi d\theta$).

With the definition of the spacetime interval (1), in spherical coordinate system, the interval can be expressed as given in Schwarzschild (1916); Mughal et al. (2021); Sacks and Ball (1968); Adler et al. (1965):

$$ds^2 = Adt^2 - Bdr^2 - C(r^2 d\theta^2 + r^2 sin^2\theta d\phi^2) \tag{2}$$

where

$$g_{00} = A(r), \qquad g_{11} = -B(r), \qquad g_{22} = -r^2, \qquad g_{33} = -r^2 sin^2\theta$$

with $C = 1$, as a hyper surface with constant r and t coordinates is a two sphere (Synge and Schild, 1978; Cheng, 2009; Pössel, 2020).

Thus we have, the invariant interval in case of a centrally symmetric static, stationary source in spherical coordinates:

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2 \tag{3}$$

where $A(r)$ and $B(r)$ are arbitrary functions of r only.

It is obvious that in the general ansatz for a metric in spherical coordinates, two functions emerge out as arbitrary. By considering the general expression of an invariant interval in spherical coordinates, we see that the metric components g_{00} and g_{11} has to attain certain values which are not equal to unity near to the source and become unity at infinite distance from the source which resembles Minkowski spacetime.
This shows that metric components merely have to posses a generalized expression as well, which become Minkowskian progressively at far away distance where $(r \to \infty)$, known as asymptotic flatness (Schutz, 2022), that depends on the distance from the source as one of the parameters. Unlike g_{22} and g_{33}, whose values remain fixed due to the constraint of spherical geometry, which when reduced to a two sphere has to have the same metric components as well, g_{00} and g_{11} remain arbitrary due to the fact that these metric components vary with distance from the source and no other geometrical constraint otherwise is imposed upon them.

Thus, instead of asking what conditions made the two metric components arbitrary, it is intriguing to notice that it is the absence of further geometric constraints on the metric components that brought out its arbitrariness. The only geometric constraint imposed on them was that they should very with distance from the considered field source. Any function that depends on the radial distance should be able to satisfy this condition. The goal of this paper is to investigate this possibility.

Since the Schwarzschild solution of the Einstein field equation is for a static stationary source, the metric must remain independent of time. The geometry of the spacetime considered doesn't evolve with time is another way to look at it. The metric tensor cannot depend on ϕ as well, since the geometry is spherically symmetric.

We have investigated that the metric components are time independent. Alongside our primary objective to prove the arbitrariness of the first two metric components, in this paper, we have assumed the arbitrary metric components to be functions of both (r, t). From the solution of the field equation itself, we demonstrate how the arbitrary metric components are in fact independent of time.

The vacuum solution can be obtained from the following field equation (Einstein, 1915; Baez and Bunn, 2005; Stephani et al., 2009):

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{4}$$

where R is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor.

Discussion on the Schwarzschild solution, (Cheng, 2009; Narlikar, 2002; Landau, 2013; Eddington, 1923; Chandrasekhar, 1983), in which g_{00} and g_{11} have been taken as exponential functions of r and t, while authors Kumar and Pathak (2022) consider A and B are algebraic functions of r and t.

In this work, our motive is to investigate the idea that these metric functions $A = g_{00}$ and $B = g_{11}$ are purely arbitrary and irrespective of the form of the function we assume for these metric components, we shall inevitably arrive at the final Schwarzschild solution. Most discussions on the Schwarzschild solution assume these functions to be of the exponential form, which through this work, we show that is a choice purely made to reduce the mathematical complexity involved and is of no other specific physical significance.

We investigate Schwarzschild solution for three different choice of functions in the metric (3) that appear as A and B. Since these functions are arbitrary, we investigate the arbitrariness of these functions.

2. Schwarzschild exterior solution

2.1. Trigonometric functions of r and t

In the metric (3), let us take:

$$A = \sin(\alpha) \tag{5}$$

$$B = \sin(\beta) \tag{6}$$

where α and β are functions of r and t. The invariant interval transforms into:

$$\mathrm{d}s^2 = \sin\alpha(r,t)\mathrm{d}t^2 - \sin\beta(r,t)\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2. \tag{7}$$

Metric becomes:

$$g_{\mu\nu} = \begin{pmatrix} \sin\alpha(r,t) & 0 & 0 & 0 \\ 0 & -\sin\beta(r,t) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{8}$$

$$g_{00} = \sin\alpha \qquad g_{11} = -\sin\beta \qquad g_{22} = -r^2 \qquad g_{33} = -r^2\sin^2\theta$$

$$g^{00} = \frac{1}{\sin\alpha} \qquad g^{11} = -\frac{1}{\sin\beta} \qquad g^{22} = -\frac{1}{r^2} \qquad g^{33} = -\frac{1}{r^2\sin^2\theta}.$$

2.1.1. Calculation of Christoffel symbols and Ricci tensor

Christoffel symbols can be obtained from:

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right). \tag{9}$$

Non-zero component of Christoffel symbols are given below, where a dot over a symbol indicates differentiation with respect to t and a prime over a symbol represents differentiation with respect to r.

$$\Gamma^0_{00} = \frac{1}{2}\dot{\alpha}\cot(\alpha) \qquad \Gamma^0_{11} = \frac{1}{2}\dot{\beta}\csc(\alpha)\cos(\beta) \qquad \Gamma^0_{10} = \frac{1}{2}\alpha'\cot(\alpha) = \Gamma^0_{01}$$

$$\Gamma^1_{11} = \frac{1}{2}\beta'\cot(\beta) \qquad \Gamma^1_{00} = \frac{1}{2}\alpha'\cos(\alpha)\csc(\beta) \qquad \Gamma^1_{10} = \frac{1}{2}\dot{\beta}\cot(\beta) = \Gamma^1_{01}$$

$$\Gamma^1_{22} = -r\csc(\beta) \qquad \Gamma^1_{33} = -r\sin^2(\theta)\csc(\beta) \qquad \Gamma^2_{33} = -\sin(\theta)\cos(\theta)$$

$$\Gamma^3_{23} = \cot(\theta) \qquad \Gamma^2_{12} = \frac{1}{r} \qquad \Gamma^3_{13} = \frac{1}{r}.$$

Components of the Ricci tensor can be obtained from the relation:

$$R_{ik} = \frac{\partial\Gamma^l_{ik}}{\partial x^l} - \frac{\partial\Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}. \tag{10}$$

Thus, non-zero components of the Ricci tensor are given as:

$$\begin{aligned} R_{00} = {} & \frac{\dot{\alpha}\dot{\beta}\cot(\alpha)\cot(\beta)}{4} - \frac{\alpha'\beta'\cos(\alpha)\cot(\beta)\csc(\beta)}{4} - \frac{3\alpha'^2\csc(\alpha)\csc(\beta)}{8} \\ & + \frac{\alpha'^2\cos(2\alpha)\csc(\alpha)\csc(\beta)}{8} + \frac{\alpha'\cos(\alpha)\csc(\beta)}{r} + \frac{\alpha''\cos(\alpha)\csc(\beta)}{2} \\ & - \frac{\dot{\beta}^2\cot^2(\beta)}{4} - \frac{\ddot{\beta}\cot(\beta)}{2} + \frac{\dot{\beta}^2\csc^2(\beta)}{2} \end{aligned} \tag{11}$$

$$\begin{aligned} R_{11} = {} & \frac{\alpha'\beta'\cot(\alpha)\cot(\beta)}{4} - \frac{\dot{\alpha}\dot{\beta}\cot(\alpha)\csc(\alpha)\cos(\beta)}{4} - \frac{\alpha'^2\cot^2(\alpha)}{4} \\ & - \frac{\alpha''\cot(\alpha)}{2} + \frac{\alpha'^2\csc^2(\alpha)}{2} + \frac{\ddot{\beta}\csc(\alpha)\cos(\beta)}{2} \\ & - \frac{\dot{\beta}^2\csc(\alpha)\sin(\beta)}{2} - \frac{\dot{\beta}^2\csc(\alpha)\cos(\beta)\cot(\beta)}{4} + \frac{\beta'\cot(\beta)}{r} \end{aligned} \tag{12}$$

$$R_{22} = -\frac{r\alpha'\cot(\alpha)\csc(\beta)}{2} + \frac{r\beta'\cot(\beta)\csc(\beta)}{2} - \csc(\beta) + 1 \tag{13}$$

$$R_{33} = \sin^2(\theta)\left[-\frac{r\alpha'\cot(\alpha)\csc(\beta)}{2} + \frac{r\beta'\cot(\beta)\csc(\beta)}{2} - \csc(\beta) + 1\right] \tag{14}$$

$$R_{01} = R_{10} = \frac{\dot{\beta}\cot(\beta)}{r}. \tag{15}$$

Ricci Scalar (scalar curvature) R is given by:

$$R = g^{im} R_{im}. \tag{16}$$

Ricci Scalar:

$$\begin{aligned} R = & -\frac{3\alpha'^2 \csc^2(\alpha)\csc(\beta)}{4} + \frac{\alpha'^2 \cos(2\alpha)\csc^2(\alpha)\csc(\beta)}{4} + \frac{2\alpha'\cot(\alpha)\csc(\beta)}{r} \\ & \frac{2\csc(\beta)}{r^2} - \frac{\alpha'\beta'\cot(\alpha)\cot(\beta)\csc(\beta)}{2} + \frac{\dot{\alpha}\dot{\beta}\cot(\alpha)\csc(\alpha)\cot(\beta)}{2} \\ & + \alpha''\cot(\alpha)\csc(\beta) + \frac{\ddot{\beta}^2\csc(\alpha)\csc^2(\beta)}{2} + \frac{\dot{\beta}^2\csc(\alpha)}{2} \\ & - \ddot{\beta}\csc(\alpha)\cot(\beta) - \frac{2\beta'\cot(\beta)\csc(\beta)}{r} - \frac{2}{r^2}. \end{aligned} \tag{17}$$

Vacuum solutions of the Einstein field equation are:

$$\frac{\sin(\alpha)\left[1 - \csc(\beta) + r\beta'\cot(\beta)\csc(\beta)\right]}{r^2} = 0 \tag{18}$$

$$\frac{r\alpha'\cot(\alpha) - \sin(\beta) + 1}{r^2} = 0 \tag{19}$$

$$\frac{\dot{\beta}\cot(\beta)}{r} = 0 \tag{20}$$

$$\begin{aligned} & -\frac{3r^2\alpha'^2\csc^2(\alpha)\csc(\beta)}{8} + \frac{r^2\alpha''\cot(\alpha)\csc(\beta)}{2} + \frac{r^2\dot{\beta}^2\csc(\alpha)\csc^2(\beta)}{4} \\ & - \frac{r^2\alpha'\beta'\cot(\alpha)\cot(\beta)\csc(\beta)}{4} + \frac{r^2\dot{\alpha}\dot{\beta}\cot(\alpha)\csc(\alpha)\cot(\beta)}{4} \\ & + \frac{r^2\dot{\beta}^2\csc(\alpha)}{4} - \frac{r^2\ddot{\beta}\csc(\alpha)\cot(\beta)}{2} + \frac{r\alpha'\cot(\alpha)\csc(\beta)}{2} \\ & \frac{r^2\alpha'^2\cos(2\alpha)\csc^2(\alpha)\csc(\beta)}{8} - \frac{r\beta'\cot(\beta)\csc(\beta)}{2} = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} & \sin^2\theta\left[-\frac{r^2\alpha'\beta'\cot(\alpha)\cot(\beta)\csc(\beta)}{4} + \frac{r^2\dot{\alpha}\dot{\beta}\cot(\alpha)\csc(\alpha)\cot(\beta)}{4}\right. \\ & -\frac{3r^2\alpha'^2\csc^2(\alpha)\csc(\beta)}{8} + \frac{r^2\alpha''\cot(\alpha)\csc(\beta)}{2} + \frac{r^2\dot{\beta}^2\csc(\alpha)\csc^2(\beta)}{4} \\ & + \frac{r^2\dot{\beta}^2\csc(\alpha)}{4} - \frac{r^2\ddot{\beta}\csc(\alpha)\cot(\beta)}{2} + \frac{r\alpha'\cot(\alpha)\csc(\beta)}{2} \\ & \left.\frac{r^2\alpha'^2\cos(2\alpha)\csc^2(\alpha)\csc(\beta)}{8} - \frac{r\beta'\cot(\beta)\csc(\beta)}{2}\right] = 0. \end{aligned} \tag{22}$$

From (20) we can perceive that since $cot(\beta)$ cannot be zero, $\dot{\beta} = 0$, implying that β is equal to some constant with respect to time. This shows that β is in fact independent of the time. Since we shall use the expression for β to calculate α, α shall also turn out to be independent of the time coordinate. (21) and (22) are equivalent and so, we use (18) and (19) to calculate the exterior solution.

From (18), we get:

$$\sin\beta = \left(1 + \frac{C}{r}\right)^{-1} \tag{23}$$

where C is a constant. Using (23) and (19), we obtain:

$$\sin\alpha = \left(1 + \frac{C}{r}\right). \tag{24}$$

Substituting the values of $\sin\alpha$ and $\sin\beta$ values in (7) we obtain:

$$\mathrm{d}s^2 = \left(1 + \frac{C}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{C}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{25}$$

which is our required Schwarzschild exterior solution.

2.2. Hyperbolic functions of r and t

Similarly, now considering the constant A as and B as hyperbolic functions, let:

$$A = \sinh(\alpha) \tag{26}$$

$$B = \sinh(\beta) \tag{27}$$

where, α and β are functions of radial distance r and time t.
Invariant interval can be written as:

$$\mathrm{d}s^2 = \sinh\alpha(r,t)\mathrm{d}t^2 - \sinh\beta(r,t)\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2. \tag{28}$$

Metric becomes:

$$g_{\mu\nu} = \begin{pmatrix} \sinh\alpha(r,t) & 0 & 0 & 0 \\ 0 & -\sinh\beta(r,t) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{29}$$

$$g_{00} = \sinh\alpha \qquad g_{11} = -\sinh\beta \qquad g_{22} = -r^2 \qquad g_{33} = -r^2\sin^2\theta$$

$$g^{00} = \frac{1}{\sinh\alpha} \qquad g^{11} = \frac{-1}{\sinh\beta} \qquad g^{22} = \frac{-1}{r^2} \qquad g^{33} = \frac{-1}{r^2\sin^2\theta}.$$

2.2.1. Calculation of Christoffel symbols and Ricci tensor

From (9), non-zero components of the Christoffel symbols are:

$$\Gamma^0_{00} = \frac{1}{2}\dot{\alpha}\coth(\alpha) \qquad \Gamma^0_{11} = \frac{1}{2}\dot{\beta}\mathrm{csch}(\alpha)\cosh(\beta) \qquad \Gamma^0_{10} = \frac{1}{2}\alpha'\coth(\alpha) = \Gamma^0_{01}$$

$$\Gamma^1_{11} = \frac{1}{2}\beta'\coth(\beta) \qquad \Gamma^1_{00} = \frac{1}{2}\alpha'\cosh(\alpha)\mathrm{csch}(\beta) \qquad \Gamma^1_{10} = \frac{1}{2}\dot{\beta}\coth(\beta) = \Gamma^1_{01}$$

$$\Gamma^1_{22} = -r\mathrm{csch}(\beta) \qquad \Gamma^1_{33} = -r\sin^2(\theta)\mathrm{csch}(\beta) \qquad \Gamma^2_{33} = -\sin(\theta)\cos(\theta)$$

$$\Gamma^3_{23} = \cot(\theta) \qquad \Gamma^2_{12} = \frac{1}{r} \qquad \Gamma^3_{13} = \frac{1}{r}.$$

Using (10) non-zero components of Ricci tensor are:

$$\begin{aligned} R_{00} = {} & \frac{\dot{\alpha}\dot{\beta}\coth(\alpha)\coth(\beta)}{4} - \frac{\alpha'\beta'\cosh(\alpha)\coth(\beta)\mathrm{csch}(\beta)}{4} - \frac{3\alpha'^2\mathrm{csch}(\alpha)\mathrm{csch}(\beta)}{8} \\ & + \frac{\alpha'^2\cosh(2\alpha)\mathrm{csch}(\alpha)\mathrm{csch}(\beta)}{8} + \frac{\alpha'\cosh(\alpha)\mathrm{csch}(\beta)}{r} + \frac{\alpha''\cosh(\alpha)\mathrm{csch}(\beta)}{2} \\ & - \frac{\dot{\beta}^2\coth^2(\beta)}{4} - \frac{\ddot{\beta}\coth(\beta)}{2} + \frac{\dot{\beta}^2\mathrm{csch}^2(\beta)}{2} \end{aligned} \tag{30}$$

$$\begin{aligned} R_{11} = {} & \frac{\alpha'\beta'\coth(\alpha)\coth(\beta)}{4} - \frac{\dot{\alpha}\dot{\beta}\coth(\alpha)\mathrm{csch}(\alpha)\cosh(\beta)}{4} - \frac{\alpha''\coth(\alpha)}{2} \\ & + \frac{3\alpha'^2\mathrm{csch}^2(\alpha)}{8} - \frac{\alpha'^2\cosh(2\alpha)\mathrm{csch}^2(\alpha)}{8} - \frac{3\dot{\beta}^2\mathrm{csch}(\alpha)\mathrm{csch}(\beta)}{8} \\ & + \frac{\dot{\beta}^2\mathrm{csch}(\alpha)\cosh(2\beta)\mathrm{csch}(\beta)}{8} + \frac{\ddot{\beta}\mathrm{csch}(\alpha)\sinh(2\beta)\mathrm{csch}(\beta)}{4} + \frac{\beta'\coth(\beta)}{r} \end{aligned} \tag{31}$$

$$R_{22} = -\frac{r\alpha'\coth(\alpha)\mathrm{csch}(\beta)}{2} + \frac{r\beta'\coth(\beta)\mathrm{csch}(\beta)}{2} - \mathrm{csch}(\beta) - 1 \tag{32}$$

$$R_{33} = \sinh^2(\theta)\left[-\frac{r\alpha'\coth(\alpha)\mathrm{csch}(\beta)}{2} + \frac{r\beta'\coth(\beta)\mathrm{csch}(\beta)}{2} - \mathrm{csch}(\beta) - 1\right] \tag{33}$$

$$R_{01} = R_{10} = \frac{\dot{\beta}\coth(\beta)}{r}. \tag{34}$$

Scalar curvature R based on (16):

$$\begin{aligned} R = {} & \frac{2\mathrm{csch}(\beta)}{r^2} - \frac{\alpha'\beta'\coth(\alpha)\coth(\beta)\mathrm{csch}(\beta)}{2} + \frac{\dot{\alpha}\dot{\beta}\coth(\alpha)\mathrm{csch}(\alpha)\coth(\beta)}{2} \\ & - \frac{3\alpha'^2\mathrm{csch}^2(\alpha)\mathrm{csch}(\beta)}{4} + \frac{\alpha'^2\cosh(2\alpha)\mathrm{csch}^2(\alpha)\mathrm{csch}(\beta)}{4} + \frac{2\alpha'\coth(\alpha)\mathrm{csch}(\beta)}{r} \\ & - \ddot{\beta}\mathrm{csch}(\alpha)\coth(\beta) - \frac{\dot{\beta}^2\mathrm{csch}(\alpha)\coth^2(\beta)}{4} - \frac{2\beta'\coth(\beta)\mathrm{csch}(\beta)}{r} \\ & + \alpha''\coth(\alpha)\mathrm{csch}(\beta) + \frac{3\dot{\beta}^2\mathrm{csch}(\alpha)\mathrm{csch}^2(\beta)}{4} - \frac{\dot{\beta}^2\mathrm{csch}(\alpha)}{4} + \frac{2}{r^2}. \end{aligned} \tag{35}$$

Vacuum solution of Einstein field equation:

$$\frac{\sinh(\alpha)\left[1-\operatorname{csch}(\beta)+r\beta'\coth(\beta)\operatorname{csch}(\beta)\right]}{r^2}=0 \tag{36}$$

$$\begin{aligned}
&-\frac{3r^2\alpha'^2\operatorname{csch}^2(\alpha)\operatorname{csch}(\beta)}{8}+\frac{r^2\alpha''\coth(\alpha)\operatorname{csch}(\beta)}{2}+\frac{r^2\alpha'^2\cosh(2\alpha)\operatorname{csch}^2(\alpha)\operatorname{csch}(\beta)}{8}\\
&-\frac{r^2\alpha'\beta'\coth(\alpha)\coth(\beta)\operatorname{csch}(\beta)}{4}-\frac{3r^2\dot\beta^2\operatorname{csch}(\alpha)\operatorname{csch}^2(\beta)}{8}+\frac{r\alpha'\coth(\alpha)\operatorname{csch}(\beta)}{2}\\
&\qquad+\frac{r^2\dot\beta^2\operatorname{csch}(\alpha)\cosh(2\beta)\operatorname{csch}^2(\beta)}{8}+\frac{r^2\ddot\beta\operatorname{csch}(\alpha)\sinh(2\beta)\operatorname{csch}^2(\beta)}{4}\\
&\qquad-\frac{r^2\dot\alpha\dot\beta\coth(\alpha)\operatorname{csch}(\alpha)\coth(\beta)}{4}-\frac{r\beta'\coth(\beta)\operatorname{csch}(\beta)}{2}=0
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\sin^2\theta\left[-\frac{3r^2\alpha'^2\operatorname{csch}^2(\alpha)\operatorname{csch}(\beta)}{8}+\frac{r^2\alpha''\coth(\alpha)\operatorname{csch}(\beta)}{2}+\frac{r^2\alpha'^2\cosh(2\alpha)\operatorname{csch}^2(\alpha)\operatorname{csch}(\beta)}{8}\right.\\
&\quad-\frac{r^2\alpha'\beta'\coth(\alpha)\coth(\beta)\operatorname{csch}(\beta)}{4}-\frac{3r^2\dot\beta^2\operatorname{csch}(\alpha)\operatorname{csch}^2(\beta)}{8}+\frac{r\alpha'\coth(\alpha)\operatorname{csch}(\beta)}{2}\\
&\qquad+\frac{r^2\dot\beta^2\operatorname{csch}(\alpha)\cosh(2\beta)\operatorname{csch}^2(\beta)}{8}+\frac{r^2\ddot\beta\operatorname{csch}(\alpha)\sinh(2\beta)\operatorname{csch}^2(\beta)}{4}\\
&\qquad\left.-\frac{r^2\dot\alpha\dot\beta\coth(\alpha)\operatorname{csch}(\alpha)\coth(\beta)}{4}-\frac{r\beta'\coth(\beta)\operatorname{csch}(\beta)}{2}\right]=0
\end{aligned} \tag{38}$$

$$\frac{r\alpha'\coth(\alpha)-\sinh(\beta)+1}{r^2}=0 \tag{39}$$

$$\frac{\dot\beta\coth(\beta)}{r}=0. \tag{40}$$

Hence from (40), as $\coth\beta\neq 0$; $\dot\beta=0$ i.e. β is independent of time.

Ex(36) gives:

$$\sinh\beta=\left(1+\frac{C}{r}\right)^{-1} \tag{41}$$

where C is a constant. Using (41) in (39) we have:

$$\sinh\alpha=\left(1+\frac{C}{r}\right). \tag{42}$$

Ex(28) with (42) and (41) is:

$$ds^2=\left(1+\frac{C}{r}\right)dt^2-\left(1+\frac{C}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{43}$$

2.3. Logarithmic functions of r and t

Considering

$$A=\ln\alpha \tag{44}$$

$$B=\ln\beta \tag{45}$$

where α and β are functions of both t and r.

Invariant interval is:

$$ds^2=\ln[\alpha(r,t)]dt^2-\ln[\beta(r,t)]dr^2-r^2d\theta^2-r^2\sin^2(\theta)d\phi^2. \tag{46}$$

Metric:

$$g_{\mu\nu}=\begin{pmatrix}\ln\alpha(r,t)&0&0&0\\0&-\ln\beta(r,t)&0&0\\0&0&-r^2&0\\0&0&0&-r^2\sin^2\theta\end{pmatrix} \tag{47}$$

$$g_{00}=\ln\alpha\qquad g_{11}=-\ln\beta\qquad g_{22}=-r^2\qquad g_{33}=-r^2\sin^2\theta$$

$$g^{00}=\frac{1}{\ln\alpha}\qquad g^{11}=-\frac{1}{\ln\beta}\qquad g^{22}=-\frac{1}{r^2}\qquad g^{33}=-\frac{1}{r^2\sin^2\theta}.$$

2.3.1. Calculation of Christoffel symbols and Ricci tensor

Using (9), listing non-zero components of the Christoffel Symbols:

$$\Gamma^0_{00} = \frac{\dot{\alpha}}{2\alpha \ln\alpha} \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{\alpha'}{2\alpha \ln\alpha} \qquad \Gamma^0_{11} = \frac{\dot{\beta}}{2\beta \ln\alpha}$$

$$\Gamma^1_{00} = \frac{\alpha'}{2\alpha \ln\beta} \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{\beta}}{2\beta \ln\beta} \qquad \Gamma^1_{11} = \frac{\beta'}{2\beta \ln\beta}$$

$$\Gamma^1_{22} = -\frac{r}{\ln\beta} \qquad \Gamma^1_{33} = -\frac{r\sin^2(\theta)}{\ln\beta}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^2_{33} = -\sin(\theta)\cos(\theta)$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot(\theta)$$

From (10), the non-zero Ricci tensor components:

$$\begin{aligned} R_{00} = {} & \frac{\beta'\alpha'}{4\beta\alpha \ln^2\beta} - \frac{\dot{\beta}\dot{\alpha}}{4\beta\alpha \ln\beta \ln\alpha} - \frac{\dot{\beta}^2}{4\beta^2 \ln^2\beta} - \frac{\dot{\beta}^2}{2\beta^2 \ln\beta} \\ & + \frac{\ddot{\beta}}{2\beta \ln\beta} + \frac{\alpha'^2}{2\alpha^2 \ln\beta} + \frac{\alpha'^2}{4\alpha^2 \ln\beta \ln\alpha} - \frac{\alpha''}{2\alpha \ln\beta} - \frac{\alpha'}{r\alpha \ln\beta} \end{aligned} \tag{48}$$

$$R_{01} = R_{10} = -\frac{\dot{\beta}}{r\beta \ln\beta} \tag{49}$$

$$\begin{aligned} R_{11} = {} & \frac{\dot{\beta}\dot{\alpha}}{4\beta\alpha \ln^2\alpha} - \frac{\beta'\alpha'}{4\beta\alpha \ln\beta \ln\alpha} + \frac{\dot{\beta}^2}{2\beta^2 \ln\alpha} + \frac{\dot{\beta}^2}{4\beta^2 \ln\beta \ln\alpha} \\ & - \frac{\ddot{\beta}}{2\beta \ln\alpha} - \frac{\beta'}{r\beta \ln\beta} - \frac{\alpha'^2}{4\alpha^2 \ln^2\alpha} + \frac{\alpha''}{2\alpha \ln\alpha} - \frac{\alpha'^2}{2\alpha^2 \ln\alpha} \end{aligned} \tag{50}$$

$$R_{22} = -\frac{r\beta'}{2\beta \ln^2\beta} + \frac{r\alpha'}{2\alpha \ln\beta \ln\alpha} + \frac{1}{\ln\beta} - 1 \tag{51}$$

$$R_{33} = -\frac{r\beta'\sin^2\theta}{2\beta \ln^2\beta} + \frac{r\alpha'\sin^2\theta}{2\alpha \ln\beta \ln\alpha} + \frac{\sin^2\theta}{\ln\beta} - \sin^2\theta. \tag{52}$$

Obtaining the Ricci scalar using (16),

$$\begin{aligned} R = {} & \frac{2}{r^2 \ln\beta} + \frac{\dot{\beta}\dot{\alpha}}{2\beta\alpha \ln\beta \ln^2\alpha} - \frac{\beta'\alpha'}{2\beta\alpha \ln^2\beta \ln\alpha} + \frac{\dot{\beta}^2}{2\beta^2 \ln^2\beta \ln\alpha} \\ & + \frac{\dot{\beta}^2}{\beta^2 \ln\beta \ln\alpha} - \frac{\ddot{\beta}}{\beta \ln\beta \ln\alpha} - \frac{2\beta'}{r\beta \ln^2\beta} - \frac{\alpha'^2}{2\alpha^2 \ln\beta \ln^2\alpha} \\ & + \frac{2\alpha'}{r\alpha \ln\beta \ln\alpha} + \frac{\alpha''}{\alpha \ln\beta \ln\alpha} - \frac{\alpha'^2}{\alpha^2 \ln\beta \ln\alpha} - \frac{2}{r^2}. \end{aligned} \tag{53}$$

Finally, solving the Einstein field equation (4) we get the following equations:

$$-\frac{1}{r^2 \ln\beta} + \frac{1}{r^2} + \frac{\beta'}{r\beta \ln^2\beta} = 0 \tag{54}$$

$$\frac{\dot{\beta}}{r\beta \ln\beta} = 0 \tag{55}$$

$$-\frac{\ln\beta}{r^2} + \frac{\alpha'}{r\alpha \ln\alpha} + \frac{1}{r^2} = 0 \tag{56}$$

$$\begin{aligned} & \frac{r^2\dot{\beta}\dot{\alpha}}{4\beta\alpha \ln\beta \ln^2\alpha} - \frac{r^2\beta'\alpha'}{4\beta\alpha \ln^2\beta \ln\alpha} + \frac{r^2\dot{\beta}^2}{4\beta^2 \ln^2\beta \ln\alpha} + \frac{r^2\dot{\beta}^2}{2\beta^2 \ln\beta \ln\alpha} \\ & \quad - \frac{r^2\ddot{\beta}}{2\beta \ln\beta \ln\alpha} - \frac{r^2\alpha'^2}{4\alpha^2 \ln\beta \ln^2\alpha} + \frac{r^2\alpha''}{2\alpha \ln\beta \ln\alpha} \\ & \quad - \frac{r^2\alpha'^2}{2\alpha^2 \ln\beta \ln\alpha} - \frac{r\beta'}{2\beta \ln^2\beta} + \frac{r\alpha'}{2\alpha \ln\beta \ln\alpha} = 0 \end{aligned} \tag{57}$$

$$\begin{aligned}\sin^2\theta\Bigg[&\frac{r^2\dot{\beta}\dot{\alpha}}{4\beta\alpha\ln\beta\ln^2\alpha}-\frac{r^2\beta'\alpha'}{4\beta\alpha\ln^2\beta\ln\alpha}+\frac{r^2\dot{\beta}^2}{4\beta^2\ln^2\beta\ln\alpha}+\frac{r^2\dot{\beta}^2}{2\beta^2\ln\beta\ln\alpha}\\&-\frac{r^2\ddot{\beta}}{2\beta\ln\beta\ln\alpha}-\frac{r^2\alpha'^2}{4\alpha^2\ln\beta\ln^2\alpha}+\frac{r^2\alpha''}{2\alpha\ln\beta\ln\alpha}\\&-\frac{r^2\alpha'^2}{2\alpha^2\ln\beta\ln\alpha}-\frac{r\beta'}{2\beta\ln^2\beta}+\frac{r\alpha'}{2\alpha\ln\beta\ln\alpha}\Bigg]=0.\end{aligned} \tag{58}$$

Equation (55) gives the time independence of β and from equation (54) we obtain:

$$\ln\beta=\left(1+\frac{C}{r}\right)^{-1}. \tag{59}$$

Using (59) and (56) we get:

$$\ln\alpha=\left(1+\frac{C}{r}\right) \tag{60}$$

where C is a constant of integration.

Substituting the values of $\ln\alpha$ and $\ln\beta$ in (46) we obtain:

$$\mathrm{d}s^2=\left(1+\frac{C}{r}\right)\mathrm{d}t^2-\left(1+\frac{C}{r}\right)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\theta^2-r^2\sin^2\theta\mathrm{d}\phi^2 \tag{61}$$

which is our required solution.

When $r\to\infty$, then A and $B\to 1$ in (2) (Kassner, 2017) i.e., the spacetime becomes Minkowskian.

The constant C, which can be regarded as a function of the mass of the object, is determined based on the consideration that at large distances from the object, as the spacetime flattens out, the gravitational field becomes weak in response to the reduction in curvature. At such distances from the object, Newton's law must hold true. In terms of the components of the metric tensor, $g_{00}=\left(1+\frac{2\Phi(x)}{c^2}\right)$ (Adler et al., 1965), so that at large distances, gravitational potential $\Phi(x)$ tends to zero making the component of the metric tensor g_{00}=1, approaching the Newtonian limit. This requires that the gravitational potential has it's Newtonian limit $\Phi(x)=-\frac{GM}{r}$ (Visser, 2005) where M is the total mass of the gravitating object and G is the universal gravitational constant. Hence constant in (25) is $C=-\frac{2GM}{c^2}$, where c is the speed of light. It can be obtained (Cheng, 2009; Landau, 2013; Kassner, 2017; Hartle, 2003) that the Schwarzschild radius is $r_s=\frac{2GM}{c^2}$.

The geometry beyond this Schwarzschild limit can be considered by removing the coordinate singularities through coordinate transformations (Cheng, 2009; Gsponer, 2004; Blau, 2011; Thorne et al., 2017). Heinicke and Hehl have given a detailed tabular format of Schwarzschild metric under various coordinate transformation including the characteristic properties each transformation conveys (Heinicke and Hehl, 2015).

Thus, with $C=-\frac{2GM}{c^2}$, and taking the convention $c=1$, the Schwarzschild metric (61) is:

$$\mathrm{d}s^2=\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2-\left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\theta^2-r^2\sin^2\theta\mathrm{d}\phi^2. \tag{62}$$

The Schwarzschild metric is derived with the condition that the spacetime is static, implying that it admits a time-like killing vector, due to translational symmetry within the limit $r>2M$ as beyond the Schwarzschild radius the time-like coordinate becomes space-like and the Killing vector will remain hypersurface orthogonal (Carroll, 2019; Mukherjee and Roy, 2021; Herrera and Witten, 2018)

$$K_1=\vec{e_t}. \tag{63}$$

Besides the time-like Killing vector which can be directly read of from the expression of the Schwarzschild metric (61), Schwarzschild geometry being spherically symmetric admits three more space-like Killing vector fields that arise as a consequence of invariance for rotation about the three spacial coordinate axes (Carroll, 2019; Mukherjee and Roy, 2021; Herrera and Witten, 2018).

$$K_2=\vec{e_\phi} \tag{64}$$

$$K_3=\cos\phi\,\vec{e_\theta}-\cot\theta\sin\phi\,\vec{e_\phi} \tag{65}$$

$$K_4=-\sin\phi\,\vec{e_\theta}-\cot\theta\cos\phi\,\vec{e_\phi}. \tag{66}$$

K_2 can also be easily read of from the expression (61) as it is evident that none of the metric components are functions of ϕ.

In fact, these are the only Killing vector fields for the Schwarzschild spacetime (Carroll, 2019). It indeed reveals the fact that no matter what form of function we assign initially for the arbitrary metric components $A(r)$ and $B(r)$, the Killing vector fields for the Schwarzschild spacetime remain the same. This means that the symmetry of the spacetime remains conserved no matter the choice of function we assume for the arbitrary metric components, hence providing with further confirmation of the arbitrary nature of the two metric components. The preservation of symmetry under the various choice of the function for the two arbitrary metric elements indicates that as long as the metric components remain a function of the radial coordinate only, the Schwarzschild metric is purely independent on the choice of function for $A(r)$ and $B(r)$.

3. Conclusion

From (2) the general expression for Schwarzschild metric leaves the two metric components as arbitrary functions of the radial coordinate. In this paper, our motive was to validate the fact that these metric components g_{00} and g_{11} are in fact truly arbitrary in the choice of function. The detailed analysis we followed validates our assumption that the Schwarzschild metric is truly independent of the choice of the function for the arbitrary metric elements $A(r)$ and $B(r)$. In most popular literature, these functions are initially assumed as exponential functions of r and t and are then proceeded to obtain the metric expression. In our work, our motive was to demonstrate the fact that this particular choice of exponential function for the first two metric components is in fact a choice aimed at achieving mathematical simplicity and is not based on any physical significance. We validate this by considering three other choice of functions for the first two metric components, namely trigonometric, hyperbolic and logarithmic form, and indeed obtain the final Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{67}$$

The choice of the functions we made here are random, aimed to prove that as long as the choice of functions are r dependent, no matter the form of function chosen for the two metric elements, we shall arrive at the final Schwarzschild metric. It is noteworthy that the Killing symmetries remain unchanged on the choice of the form of the functions, which indicates a lack of physical significance of the choice of function on the symmetry of spacetime, implying that the choice of function as exponential functions of r and t in the usual approach to obtain the Schwarzschild metric is aimed at mathematical simplification.

The exponential form of the function is sometimes justified by stating that it leaves the metric elements strictly positive. The fact that we are able to derive the Schwarzschild metric with other choice of functions such as trigonometric, hyperbolic and logarithmic, which can assume non-positive values reveals that the strict positive nature of the two metric elements assumed elsewhere is not a necessary requirement to obtain the Schwarzschild solution.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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