

Extended minimal theories of massive gravityAntonio De Felice,¹ Shinji Mukohyama,^{1,2} and Masroor C. Pookkillath^{1,3}¹*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan*²*Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan*³*Centre for Theoretical Physics and Natural Philosophy, Mahidol University, Nakhonsawan Campus, Phayuha Khiri, Nakhonsawan 60130, Thailand*

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In this work, we introduce a class of extended minimal theories of massive gravity, without requiring *a priori* that the theory should admit the same homogeneous and isotropic cosmological solutions as the de Rham-Gabadadze-Tolley massive gravity. The theory is constructed as to have only two degrees of freedom in the gravity sector. In order to perform this step we first introduce a precursor theory endowed with a general graviton mass term, to which, at the level of the Hamiltonian, we add two extra constraints as to remove the unwanted degrees of freedom, which otherwise would typically lead to ghosts and/or instabilities. On analyzing the number of independent constraints and the properties of tensor mode perturbations, we see that the gravitational waves are the only propagating gravitational degrees of freedom which do acquire a nontrivial mass, as expected. In order to understand how the effective gravitational force works for this theory we then investigate cosmological scalar perturbations in the presence of a pressureless fluid. We then restrict the whole class of models by imposing the following conditions at all times: (1) it is possible to define an effective gravitational constant, G_{eff} ; (2) the value G_{eff}/G_N is always finite but not always equal to unity (as to allow some nontrivial modifications of gravity, besides the massive tensorial modes); and (3) the square of mass of the graviton is always positive. These constraints automatically make also the ISW-effect contributions finite at all times. Finally we focus on a simple subclass of such theories, and show they already can give a rich and interesting phenomenology.

DOI: [10.1103/PhysRevD.106.084050](https://doi.org/10.1103/PhysRevD.106.084050)**I. INTRODUCTION**

In these last years, we have witnessed a boom for the research in gravity both from theoretical and experimental sides. In particular, the discovery of gravitational waves has paved the ground for a long research path which will lead to a deeper understanding of several new aspects of gravity [1]. On one side this will affect largely astrophysics, in particular the research aimed to understand the dynamics of the final states of stars in strong gravity regimes. On another end, a large sample from the detected gravitational waves seems to be coming from the merger of two black holes: the values for the masses of the black holes involved in these phenomena seem to be pointing either to nontrivial astrophysical sources or even to the existence of primordial black holes, which could be forming at least part of the dark matter content [2].

From an observational point of view, having a larger sample of neutron star mergers will also give us a link to cosmology, since the sources of the signals could be located in a far away galaxy, leading to a propagation of the gravitational waves over a cosmological distance [3]. In particular, this branch of the gravitational wave science

should help us understand the nature of the so-called H_0 tension [4,5]. As a matter of fact, the high redshift CMB data including Planck [6] as well as Atacama Cosmology Telescope (ACT) [7], and the late time data, SH0ES [8] do not agree with each other in the context of Λ CDM, the “de facto” standard model of gravity. This tension could point either to new physics or to some unexpected and nontrivial systematic errors in the data, and the gravitational waves discoveries should help in confirming or ruling out this last hypothesis.

If this situation is not already surprising in cosmology, still another observable in the data related to the growth of structure, the amplitude of the fluctuation S_8 during matter domination up to now, seems to be again showing poor agreement between early time data (Planck [6]) and late-time large scale structures [9,10], once more in the context of the Λ CDM model. These two tensions open up room for exploring models of the Universe beyond the Λ CDM, for example by modifying gravity at large scales [11–13]. See [5] for a review of possible solutions to the Hubble tension.

There have been several attempts to try to reconcile data and theory at the cost of introducing new degrees of freedom, which could change the dynamics of the cosmological

background and matter perturbation needed to solve the above mentioned puzzling tensions [12–16]. What is surprising though is that at local scales (e.g. solar system scales) there is no trace of such additional degrees of freedom which would be necessary to fix the cosmological issues [17]. One then needs to address how to hide existing new degrees of freedom in environments with energy scales much higher than the cosmological ones [18–20].

However, a more minimal approach, and possibly a simpler one, is to give a nonzero mass to the graviton [21]. If the mass, μ , of such a mode is small enough, i.e. comparable to the size of today's Hubble expansion rate ($\mu \simeq 10^{-33}$ eV), then for the typical energy scales present in astrophysical environments, the graviton would typically be largely ultrarelativistic avoiding in this way the constraints on μ coming from the propagation of gravitational waves, which is $\mu < 10^{-23}$ eV [22]. Even though the graviton mass is negligible at very short scales (i.e. solar system scales), on cosmological scales things could be different. In fact, the theory leading to a nonzero graviton mass could be becoming sensibly different from Λ CDM at late times, when $H \simeq \mu$. This theory could also be responsible for an apparent modified gravity behavior in cosmology which could be affecting both the background and cosmological perturbations, being able in this way to address both the above mentioned tensions [23,24].

Is this an interesting idea or nothing but a theorist-wild-dream scenario? In fact, the question of a nonzero mass for the graviton was posed a long time ago and first partially addressed by Fierz and Pauli [21]. Partially, because they studied a theory of massive gravity only in a perturbative regime, i.e. without knowing the theory in full, in any nonperturbative regime. Only quite recently, a theory of massive gravity which is totally consistent with a theoretical point of view was introduced, which is dubbed as de Rham-Gabadadze-Tolley (dRGT) theory [25,26]. This breakthrough led to an exploration of the phenomenology for such a theory, but it was realized that this model, at least in the simplest approach, could not have well-defined cosmological behavior [27,28]. By a beyond-linear-perturbation analysis around a homogeneous and isotropic background, it was found that at least one (out of five) of the graviton degrees of freedom would be a (light) ghost and as such would make dRGT lose its ability to make predictions [27].

Although this result might look disappointing, this negative result has led to several other possibilities. One of them consisted of introducing terms which break Lorentz invariance, in order to remove unwanted (unstable) degrees of freedom. Along these lines of research, a model called minimal theory of massive gravity (MTMG) was introduced as to resolve the issue of dRGT on a cosmological background [29]. In particular MTMG, by construction, removes three (out of five) graviton degrees of freedom in a nonlinear way, leaving tensor modes as only propagating degrees of freedom on any background. The theory has been proved to be

interesting and led to a nontrivial phenomenology discussed even recently in the literature [23,24,30]. Along the same lines, MTMG was extended as to have a scalar field in the gravity sector (in addition to the massive graviton) [31–33], and even to a minimal theory of bigravity (MTBG) [34]. It should be noted that in MTMG by construction we remove all gravity modes but the transverse-traceless gravitational waves. This was necessary as in Lorentz invariant massive gravity theories, although Vainshtein mechanism works at screening the extra modes on short scales, still homogeneous and isotropic manifolds, at least in dRGT, possess a nonlinear light ghost, which makes standard cosmology pathological [27].

Besides the requirement for the minimal number of propagating degrees of freedom, MTMG has been constructed so as to admit the same homogeneous and isotropic cosmological solutions as in dRGT, for which there are two branches of solutions: the self-accelerating branch and the normal branch. In the self-accelerating branch the graviton mass term acts as an effective cosmological constant that can accelerate the expansion of the universe [35] while the linear perturbations behave exactly the same as the standard Λ CDM, except that gravitational waves acquire a nonvanishing mass. Unlike dRGT, the self-accelerating branch of MTMG is free from fatal instabilities and thus provides a firm testing ground for gravitational wave physics of massive gravity. However, from the viewpoint of recent tensions in cosmology, this branch of MTMG is as good as but not better than Λ CDM. In this respect the normal branch of MTMG could perform better than Λ CDM. Indeed, in the normal branch of MTMG the scalar linear perturbations behave differently from Λ CDM.

Although the normal branch of MTMG has proved to be an interesting possibility as to try to modify gravity at large scales in a consistent and minimal way, still it had some features which were setting some theoretical and phenomenological issues. In particular, MTMG was leading to a modified effective Newtonian gravitational constant which at large scales behaves as $G_{\text{eff}}/G_N \propto (\mu^2/H^2 - 2)^{-2}$ [30]. This expression for G_{eff}/G_N is well behaved for negative-squared-mass for the graviton (for which, though, a tachyonic instability, with a timescale of order H_0^{-1} , would be affecting modes of order $k/(a_0 H_0) \simeq 1$) [36]. But for a large-enough positive-squared-mass graviton (that is still inside the allowed Ligo bounds), a range of positive μ^2 , for $\mu \simeq 2H_0$, would lead to strong modifications to G_{eff}/G_N , leading in turn to strong constraints from the data even at nonlinear scales [37]. In particular, in a recent paper, on studying the effect of Planck data on MTMG, it was discovered that positive μ^2 is actually preferred but because of the above mentioned behavior of G_{eff}/G_N , μ^2 , it is strongly constrained toward values very close to zero [24]. This phenomenon puts strong limits on the normal branch of MTMG.

In this paper, we try to solve these issues of the normal branch of MTMG by extending the MTMG itself, in a way

which is meant to cure the above mentioned behavior of G_{eff}/G_N . In order to extend MTMG we still need to add constraints to the Hamiltonian of a precursor theory as to remove the unwanted degrees of freedom, but we change the constraints themselves. One of the constraints of MTMG was chosen as to admit exactly the same cosmological background as dRGT. As mentioned above, this constraint defining MTMG was leading to the presence of two branches for the background dynamics. On the other hand, the extended MTMG (eMTMG) has in general a different background dynamics from dRGT, especially if these same modifications/extensions lead to a better behaved phenomenology. Indeed, eMTMG allows for a much larger freedom in terms of background dynamics, still being a minimal theory (i.e. with only two tensor propagating degrees of freedom on any background). However, as we shall see later on, the condition that on any allowed cosmological background G_{eff}/G_N will never have poles and μ^2 being non-negative, will considerably reduce the set of allowed theories. Still, we give a proof of existence of a large class of models which indeed satisfy these criteria (and which by construction does not reduce to dRGT at the background level). We also show that at the level of the background and linear perturbations, all predictions of the models in this class are captured by a smaller subclass of eMTMG with only six parameters, which determine the cosmological constant and the behavior of μ_0 and $G_{\text{eff},0}/G_N$. One of the parameters, c_4 , which we will see later on, is equivalent to the cosmological constant in the Λ CDM model. The other five are completely new which affects both background and perturbation.

As was happening in MTMG, for environmental densities much larger than the present cosmological ones, that is $\rho \gg M_{\text{pl}}^2 H_0^2$ (valid at solar system scales and at high redshifts) we find that $G_{\text{eff}}/G_N \rightarrow 1$.

This paper is organized as follows. Section II shows the construction of the eMTMG, where we introduce two general functions, F_1 and F_2 , for which we make use of the Cayley-Hamilton theorem. In particular, after writing down a precursor theory, we add constraints to make the theory minimal, in the sense that no additional degrees of freedom are propagating in the gravitational sector besides the gravitational waves, which become massive. Then, in Sec. III, we study the spatially flat, homogeneous and isotropic cosmological background in this theory. Here, using the minisuperspace Hamiltonian and the constraints, we show that one of the Lagrange multipliers $\lambda(t)$ vanishes in the spatially flat, homogeneous and isotropic background. Unlike the original MTMG, the space of solutions is not separated into two branches: the self-accelerating branch and the normal branch, rather there is one and only one universal branch. While in Appendix C we consider the condition under which the separation into the two branches occurs, in the rest of the present paper we study the general case. In Sec. IV, we study linear perturbations around the

spatially flat, homogeneous and isotropic background in this theory. We first consider the propagation of the gravitational waves on the cosmological background. As expected, the two modes are now massive. Subsequently, we derive the expression for the G_{eff}/G_N considering the eMTMG minimally coupled with a pressureless fluid. Furthermore, we derive equations of motion for scalar perturbations in the presence of multiple perfect fluids with general equations of state. In Sec. V we then make a list of phenomenologically motivated criteria to be imposed on the theory, which makes it possible for us to find a subset of models with a finite number of parameters. In particular, we require the finiteness at any redshift of G_{eff}/G_N which is modified in anyway at late times, i.e. without altering the early time dynamics. In addition, we impose the condition that the squared mass of the gravitational waves is positive, i.e. $\mu^2 > 0$. We also demand the finiteness of the ISW effect at any redshift. In order to give a working example for such a theory, which is nonetheless endowed with the desired features of the general model, in Sec. VI we adopt a simple polynomial ansatz for $F_{1,2}$ and impose the phenomenological criteria explained above step by step. As a consequence, we obtain a rather simple subclass of the general model which satisfies all the criteria. It turns out that

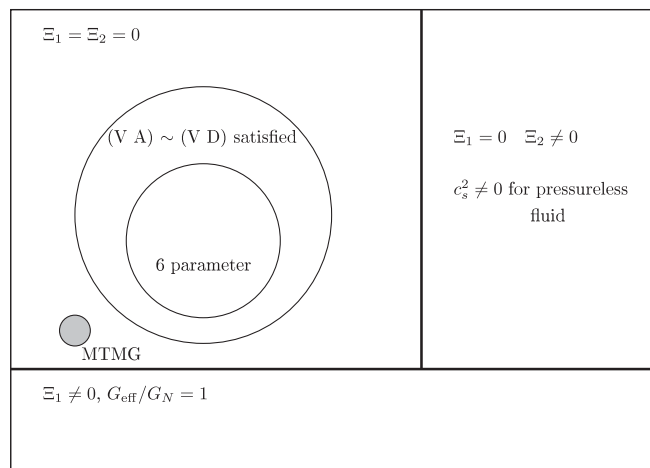


FIG. 1. This figure shows relations among different subclasses of the eMTMG and the original MTMG. The classification has been made according to two scalar quantities which determine the phenomenology of G_{eff} . Other criteria for the classification can be in principle considered. The region where phenomenological criteria (V A–V D) are satisfied determines a class of models with appealing phenomenological properties, e.g. G_{eff}/G_N is finite for any dynamics of the cosmological background, the tensor graviton has a non-negative mass squared, etc. Finally it is possible to give a simple subset (having at most six free parameters) which already possesses all the defining properties of the model. The quantities Ξ_1 and Ξ_2 are defined in Eqs. (66) and (67) respectively. The quantity Ξ_1 , which depends on the function F_2 , discriminates the behavior of the theory in the high- k regime. Also the quantity Ξ_2 , the mirror quantity for the function F_1 , has a strong influence on the phenomenology of the theory.

at the level of the background and the linear perturbations, all observables within this subclass depend only on six parameters while $F_{1,2}$ depend on more parameters. We thus remove this degeneracy by defining a further simpler subclass, by picking up the model for $F_{1,2}$ which only shows the above-mentioned six free parameters, i.e. five more than Λ CDM. Finally, we report our conclusion in Sec. VII. Figure 1 summarizes the results. We find it useful to add five appendices to the main text. Appendix A shows some useful variational formulas needed for the construction of the theory. In Appendix B we discuss the original MTMG as a special case of this eMTMG. In Appendix C we consider the condition under which the space of spatially flat, homogeneous and isotropic solutions of the eMTMG is divided into two branches, the so-called self-accelerating and normal branches. In Appendix D, we provide the full expression for G_{eff}/G_N and the ISW potential field. Finally, Appendix E discusses a rather peculiar model having massless tensor modes (on the cosmological background), with nontrivial dynamics for the scalar perturbations, i.e. $G_{\text{eff}}/G_N \neq 1$.

II. MODEL CONSTRUCTION

A. Building blocks

In order to build up the model, we will follow a path which is similar to the one followed in [29,30]. First of all, in the following, we will make use of the unitary gauge and the metric formalism.¹ In the unitary gauge we introduce a three-dimensional fiducial metric with positive definite signature, which is, by construction of the theory, an external, explicitly time (and time only) dependent field, that we denote by $\tilde{\gamma}_{ij}(t)$. In the unitary gauge we will also introduce another external field, M , that we call fiducial lapse function. In order for the theory to allow spatially flat, homogeneous and isotropic solutions, we require the fiducial sector to be compatible with the symmetry of such solutions. For this reason we will identify $\tilde{\gamma}_{ij} = \tilde{a}(t)^2 \delta_{ij}$ and $M = M(t)$, where $\tilde{a}(t)$ is the fiducial scale factor. This three-dimensional fiducial metric admits an inverse, denoted by $\tilde{\gamma}^{ij}$, which satisfies $\tilde{\gamma}^{il}\tilde{\gamma}_{lj} = \delta^i_j$. Out of these external fields, we can also define the following field $\tilde{\zeta}^i_j$ as

$$\tilde{\zeta}^i_j \equiv \frac{1}{2M} \tilde{\gamma}^{il} \dot{\tilde{\gamma}}_{lj}, \quad (1)$$

which describes the rate of change of the fiducial metric.² Notice that in the unitary gauge description, having the

¹Using the unitary gauge, although not strictly necessary, turns out to be simplifying the calculations. As for the choice of the metric formalism, one could equivalently choose the vielbein formalism to define the theory, as done in [29,30].

²In the vielbein formalism we instead define $\tilde{\zeta}^i_j = \frac{1}{M} E^i_A \dot{E}^A_j$, where $\tilde{\gamma}_{ij} = \delta_{AB} E^A_i E^B_j$, giving $\tilde{\zeta}^i_j = \frac{1}{2} (\tilde{\zeta}^i_j + \tilde{\gamma}^{il} \tilde{\zeta}^k_{il} \tilde{\gamma}_{jk})$, which in any case agree with each other when $\tilde{\gamma}_{ij} = \tilde{a}^2 \delta_{ij}$.

presence of the external fields which required a full coordinate choice, will explicitly break four-dimensional diffeomorphism, and a choice of slicing has been automatically fixed.

Of course, we also have physical, dynamical metric variables, which we adopt from the ADM formalism. In particular we have a lapse function N , a shift vector N^i and a three-dimensional metric γ_{ij} , which admits the inverse γ^{ij} . Out of them, the four-dimensional physical metric can be written as

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt). \quad (2)$$

Having introduced the dynamical field γ_{ij} and the external field $\tilde{\gamma}_{ij}$, which in unitary gauge have fixed, given dynamics, we can introduce the building blocks of the theory \mathcal{K}^i_j and \mathfrak{K}^i_j which satisfy the following properties

$$\mathcal{K}^i_l \mathcal{K}^l_j = \tilde{\gamma}^{il} \gamma_{lj}, \quad (3)$$

$$\mathfrak{K}^i_l \mathfrak{K}^l_j = \gamma^{il} \tilde{\gamma}_{lj}, \quad (4)$$

$$\mathcal{K}^i_l \mathfrak{K}^l_j = \delta^i_j = \mathfrak{K}^i_l \mathcal{K}^l_j. \quad (5)$$

Some useful formulas for the variations of the quantities defined above are summarized in Appendix A.

B. Precursor Hamiltonian

We now have all the required building blocks to define the theory, and we will do so by writing down its Hamiltonian density, and then via a Legendre transformation, we will find its Lagrangian density. Then, along the same lines of MTMG, see e.g. [30], we first introduce a precursor Hamiltonian density, which we now define to be

$$\begin{aligned} \mathcal{H}_{\text{pre}} \equiv & -N \mathcal{R}_0^{\text{GR}} - N^i \mathcal{R}_i + \frac{1}{2} m^2 M_{\text{P}}^2 N \sqrt{\gamma} F_1([\mathfrak{K}], [\mathfrak{K}^2], \\ & [\mathfrak{K}^3]) + \frac{1}{2} m^2 M_{\text{P}}^2 M \sqrt{\tilde{\gamma}} F_2([\mathcal{K}], [\mathcal{K}^2], [\mathcal{K}^3]), \end{aligned} \quad (6)$$

where

$$\mathcal{R}_0^{\text{GR}} = \frac{M_{\text{P}}^2}{2} \sqrt{\gamma} R^{(3)} - \frac{2}{M_{\text{P}}^2} \sqrt{\gamma} \left(\gamma_{ik} \gamma_{jl} - \frac{1}{2} \gamma_{ji} \gamma_{kl} \right) \tilde{\pi}^{ij} \tilde{\pi}^{lk}, \quad (7)$$

$$\mathcal{R}_i = 2\sqrt{\tilde{\gamma}} \gamma_{ij} D_k \tilde{\pi}^{jk}, \quad (8)$$

$$\tilde{\pi}^{ij} \equiv \frac{\pi^{ij}}{\sqrt{\gamma}}, \quad (9)$$

and $[\mathcal{K}] \equiv \mathcal{K}^i_i$, $[\mathcal{K}^2] \equiv \mathcal{K}^i_j \mathcal{K}^j_i$, etc. Here we point out that the fields N and N^i have been considered to be Lagrange multipliers, whereas the dynamical degrees of freedom enter in the six independent components of γ_{ij} , which lead, in turn, to twelve phase space variables, since π^{ij}

correspond to their conjugate momenta. Here the operator D_i represents the covariant derivative compatible with the three-dimensional metric γ_{ij} .

By looking at Eq. (6), the precursor theory is defined in terms of two functions $F_{1,2}$, which depend on the trace of powers of the above defined building blocks \mathcal{K}^i_j and \mathfrak{R}^i_j . Making use of the Cayley-Hamilton theorem applied to a three-dimensional matrix, e.g. \mathfrak{R}^i_j , we only choose $[\mathfrak{R}]$, $[\mathfrak{R}^2]$, $[\mathfrak{R}^3]$ as the variables out of which the function F_1 depends on. Also by the same theorem, the mirror variables $[\mathcal{K}]$, $[\mathcal{K}^2]$, $[\mathcal{K}^3]$ can be rewritten in terms of the previous $[\mathfrak{R}]$, $[\mathfrak{R}^2]$, $[\mathfrak{R}^3]$ variables, which become the really independent ones. Therefore, on looking at the precursor Hamiltonian, we can further define the following two quantities

$$\mathcal{R}_0 \equiv \mathcal{R}_0^{\text{GR}} - \frac{1}{2} m^2 M_{\text{P}}^2 \sqrt{\tilde{\gamma}} F_1([\mathfrak{R}], [\mathfrak{R}^2], [\mathfrak{R}^3]), \quad (10)$$

$$H_1 = \frac{1}{2} m^2 M_{\text{P}}^2 \int d^3x M \sqrt{\tilde{\gamma}} F_2([\mathcal{K}], [\mathcal{K}^2], [\mathcal{K}^3]). \quad (11)$$

Indeed, for this precursor theory, the four Lagrange multipliers N and N^i set four constraints, whereas H_1 corresponds to the Hamiltonian of the precursor theory evaluated on the constraint surface (on which \mathcal{R}_0 and \mathcal{R}_i all vanish). One can then evaluate the time derivative of the constraints \mathcal{R}_0 and \mathcal{R}_i . As for $\dot{\mathcal{R}}_0$, we would find $\dot{\mathcal{R}}_0 = -N^i \{\mathcal{R}_0, \mathcal{R}_i\} + \dots$, which needs to vanish on the constraints surface. However the Poisson brackets $\{\mathcal{R}_0, \mathcal{R}_i\}$ do not all vanish, then setting $\dot{\mathcal{R}}_0 \approx 0$, would actually fix one of the Lagrange multipliers without imposing any new constraint on the theory. Indeed,

since the rank of $\{\mathcal{R}_0, \mathcal{R}_i\}$ is two, not all the eight $\mathcal{R}_0, \mathcal{R}_i, \dot{\mathcal{R}}_0, \dot{\mathcal{R}}_i$ are constraints, but only six of them. This means that this theory has $\frac{1}{2}(12 - 6) = 3$ degrees of freedom, where twelve represents the number of independent components of γ_{ij} and their conjugate momenta in the phase space.

The precursor theory we have considered consists of three degrees of freedom. These are the two polarizations of the transverse-traceless gravitational waves and the other one is an additional mode: it is a scalar on a homogeneous and isotropic manifold. Our idea, to be studied in detail in the next subsection, is to remove the scalar mode by adding additional constraints to the precursor theory.

By using the Stückelberg trick, we will be seeing that one of them (or better a gauge invariant combination of fields) corresponds to the extra scalar mode which we want to remove. We shall not study the properties of this extra mode since it does not exist in the minimal theories.

C. Hamiltonian of the extended minimal theory of massive gravity

From what we have learned in the previous section, we still need to add two new constraints as to make the theory minimal, i.e. having only two propagating degrees of freedom in the gravity sector. In order to achieve this goal, we can follow the same steps of MTMG as to make the theory minimal. Let us use the on-shell precursor Hamiltonian H_1 as to define the quantities \mathcal{C}_0 and \mathcal{C}_i as follows. They would correspond to time derivatives of the \mathcal{R}_0 and \mathcal{R}_i constraints if H_1 were the Hamiltonian of the system.

$$\begin{aligned} \mathcal{C}_0 &\equiv \{\mathcal{R}_0, H_1\} + \frac{\partial \mathcal{R}_0}{\partial t} \\ &= m^2 M \sqrt{\tilde{\gamma}} (2\tilde{\gamma}^{ij}\tilde{\pi}^{ce} - \tilde{\gamma}^{ic}\tilde{\pi}^{je}) \left[\frac{1}{2} F_{2,[\mathcal{K}]} \mathfrak{R}^l{}_i \gamma_{lc} \gamma_{je} + F_{2,[\mathcal{K}^2]} \gamma_{ic} \gamma_{je} + \frac{3}{2} F_{2,[\mathcal{K}^3]} \mathcal{K}^l{}_i \gamma_{lc} \gamma_{je} \right] \\ &\quad - m^2 M_{\text{P}}^2 \sqrt{\tilde{\gamma}} M \tilde{\zeta}^i{}_j \left[\frac{1}{2} F_{1,[\mathfrak{R}]} \mathcal{K}^l{}_k \tilde{\gamma}_{li} \gamma^{kj} + F_{1,[\mathfrak{R}^2]} \tilde{\gamma}_{il} \gamma^{jl} + \frac{3}{2} F_{1,[\mathfrak{R}^3]} \mathfrak{R}^j{}_i \tilde{\gamma}_{ik} \gamma^{lk} \right], \end{aligned} \quad (12)$$

$$\begin{aligned} \mathcal{C}_i[v^i] &= \int d^3x \mathcal{C}_i v^i \equiv \{\mathcal{R}_i[v^i], H_1\} = \left\{ \int d^3x \mathcal{R}_i v^i, H_1 \right\} \\ &= \frac{1}{2} m^2 M_{\text{P}}^2 \int d^3x \sqrt{\tilde{\gamma}} D_j v^i \left[\frac{1}{2} M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} F_{2,[\mathcal{K}]} (\mathfrak{R}^j{}_i \tilde{\gamma}^{lk} \gamma_{ik} + \mathfrak{R}^k{}_i \tilde{\gamma}^{lj} \gamma_{ki}) \right. \\ &\quad \left. + 2M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} F_{2,[\mathcal{K}^2]} \gamma_{il} \tilde{\gamma}^{jl} \frac{3}{2} M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} F_{2,[\mathcal{K}^3]} (\mathcal{K}^j{}_i \tilde{\gamma}^{lk} \gamma_{ik} + \mathcal{K}^k{}_i \tilde{\gamma}^{lj} \gamma_{ki}) \right], \end{aligned} \quad (13)$$

where we have taken into consideration the fact that having chosen the unitary gauge, the constraint \mathcal{R}_0 explicitly depends on time. The notation $F_{,X}$ represents partial derivative of F with respect to X . The previous relations lead to

$$\mathcal{C}_0 = \frac{1}{2} m^2 M_{\text{P}}^2 M \sqrt{\tilde{\gamma}} (2\tilde{\gamma}^{bd} \tilde{\pi}^{ce} - \tilde{\gamma}^{bc} \tilde{\pi}^{de}) [F_{2, [\mathcal{K}]} \mathfrak{R}^a{}_b \gamma_{ac} \gamma_{de} + 2F_{2, [\mathcal{K}^2]} \gamma_{bc} \gamma_{de} + 3F_{2, [\mathcal{K}^3]} \mathcal{K}^a{}_b \gamma_{ac} \gamma_{de}] + \sqrt{\tilde{\gamma}} \mathcal{C}_\zeta, \quad (14)$$

$$\mathcal{C}_\zeta \equiv -\frac{1}{2} m^2 M_{\text{P}}^2 M \tilde{\zeta}^c{}_d (F_{1, [\mathfrak{R}]} \gamma^{db} \mathcal{K}^a{}_b \tilde{\gamma}_{ac} + 2F_{1, [\mathfrak{R}^2]} \gamma^{db} \tilde{\gamma}_{bc} + 3F_{1, [\mathfrak{R}^3]} \mathfrak{R}^d{}_b \gamma^{be} \tilde{\gamma}_{ec}), \quad (15)$$

$$\mathcal{C}_i = -m^2 M_{\text{P}}^2 \sqrt{\tilde{\gamma}} D_j \left\{ M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} \left[\frac{F_{2, [\mathcal{K}]}}{4} (\mathfrak{R}^j{}_i \tilde{\gamma}^{lk} + \mathfrak{R}^k{}_i \tilde{\gamma}^{lj}) \gamma_{ki} + F_{2, [\mathcal{K}^2]} \tilde{\gamma}^{jl} \gamma_{il} + \frac{3}{4} F_{2, [\mathcal{K}^3]} (\mathcal{K}^j{}_i \tilde{\gamma}^{lk} + \mathcal{K}^k{}_i \tilde{\gamma}^{lj}) \gamma_{ki} \right] \right\}. \quad (16)$$

We are now ready to define the extended MTMG theory by giving its Hamiltonian density as

$$\mathcal{H} = -N \mathcal{R}_0 - N^i \mathcal{R}_i + \frac{1}{2} m^2 M_{\text{P}}^2 M \sqrt{\tilde{\gamma}} F_2([\mathcal{K}], [\mathcal{K}^2], [\mathcal{K}^3]) - \lambda \mathcal{C}_0 - \lambda^i \mathcal{C}_i. \quad (17)$$

Now all the eight constraints, imposed by the Lagrange multipliers N , N^i , λ , λ^i , are second class which then leave only two dynamical degrees of freedom. We can write down the Hamiltonian of the theory as

$$H = \int d^3x \left[-N \mathcal{R}_0 - N^a \mathcal{R}_a + \frac{1}{2} m^2 M_{\text{P}}^2 M \sqrt{\tilde{\gamma}} F_2([\mathcal{K}], [\mathcal{K}^2], [\mathcal{K}^3]) - \lambda \mathcal{C}_0 - \sqrt{\tilde{\gamma}} (D_j \lambda^i) \mathcal{C}_i^j \right], \quad (18)$$

where we have introduced the three-dimensional tensor

$$\mathcal{C}_i^j \equiv \frac{1}{2} m^2 M_{\text{P}}^2 M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} \left[\frac{1}{2} F_{2, [\mathcal{K}]} (\mathfrak{R}^j{}_k \tilde{\gamma}^{kl} + \tilde{\gamma}^{jk} \mathfrak{R}^l{}_k) \gamma_{li} + 2F_{2, [\mathcal{K}^2]} \tilde{\gamma}^{jk} \gamma_{ki} + \frac{3}{2} F_{2, [\mathcal{K}^3]} (\mathcal{K}^j{}_k \tilde{\gamma}^{kl} + \tilde{\gamma}^{jk} \mathcal{K}^l{}_k) \gamma_{li} \right]. \quad (19)$$

In summary, since the constraints for the theory now add to eight, the theory is minimal, i.e. the number of gravitational degrees of freedom is now $\frac{1}{2}(12 - 8) = 2$.

Now we have arrived at the Hamiltonian of the eMTMG, where we have removed the additional mode by adding constraint $\lambda \mathcal{C}_0$ and $\sqrt{\tilde{\gamma}} (D_j \lambda^i) \mathcal{C}_i^j$ to the precursor theory (actually only two of them are new to the precursor theory). Then the remaining two modes are independent polarization of transverse-traceless gravitational waves with mass.

D. Minimal theory Lagrangian

In order to find the Lagrangian density of the theory, we need to perform a Legendre transformation. From the Hamiltonian equations of motion for γ_{ij} , we find

$$\begin{aligned} \dot{\gamma}_{ij} = \{\gamma_{ij}, H_{\text{tot}}\} &= \frac{2N}{M_{\text{P}}^2} (2\gamma_{ik} \gamma_{jd} - \gamma_{ij} \gamma_{kd}) \tilde{\pi}^{kd} + \gamma_{ik} D_j N^k + \gamma_{jk} D_i N^k \\ &+ \frac{1}{2} m^2 \lambda M \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} [2\tilde{\gamma}^{kd} F_{2, [\mathcal{K}^2]} (\gamma_{ij} \gamma_{kd} - 2\gamma_{ik} \gamma_{jd}) - (\mathfrak{R}^k{}_d F_{2, [\mathcal{K}]} + 3\mathcal{K}^k{}_d F_{2, [\mathcal{K}^3]}) \tilde{\gamma}^{de} (\gamma_{ie} \gamma_{jk} + \gamma_{ik} \gamma_{je} - \gamma_{ij} \gamma_{ke})], \end{aligned} \quad (20)$$

so that we can also find the relation between the extrinsic curvature $K_{ij} \equiv \frac{1}{2N} (\dot{\gamma}_{ij} - \gamma_{ik} D_j N^k - \gamma_{jk} D_i N^k)$ and the canonical momenta π^{ij} as

$$\begin{aligned} K_{ij} &= \frac{1}{M_{\text{P}}^2} (2\gamma_{ik} \gamma_{jd} - \gamma_{ij} \gamma_{kd}) \tilde{\pi}^{kd} + \frac{m^2}{4} \lambda \frac{M}{N} \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} [2\tilde{\gamma}^{kd} F_{2, [\mathcal{K}^2]} (\gamma_{ij} \gamma_{kd} - 2\gamma_{ik} \gamma_{jd}) \\ &- (\mathfrak{R}^k{}_d F_{2, [\mathcal{K}]} + 3\mathcal{K}^k{}_d F_{2, [\mathcal{K}^3]}) \tilde{\gamma}^{de} (\gamma_{ie} \gamma_{jk} + \gamma_{ik} \gamma_{je} - \gamma_{ij} \gamma_{ke})], \end{aligned} \quad (21)$$

out of which we have

$$\tilde{\pi}^{ij} = \frac{M_{\text{P}}^2}{2} (\gamma^{ik} \gamma^{jd} - \gamma^{ij} \gamma^{kd}) K_{kd} - \frac{m^2 M_{\text{P}}^2 M}{8} \frac{\lambda}{N} \lambda^{\ominus ij}. \quad (22)$$

Here, we have introduced the following tensor

$$\Theta^{ij} = -\frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} [(\tilde{\gamma}^{jk} \mathfrak{R}^i_k + \tilde{\gamma}^{ik} \mathfrak{R}^j_k) F_{2, [\mathcal{K}]} + 4\tilde{\gamma}^{ij} F_{2, [\mathcal{K}^2]} + 3(\tilde{\gamma}^{jk} \mathcal{K}^i_k + \tilde{\gamma}^{ik} \mathcal{K}^j_k) F_{2, [\mathcal{K}^3]}]. \quad (23)$$

After a straightforward calculation, we can write down the Lagrangian density of the extended MTMG as

$$\begin{aligned} \mathcal{L} = & \frac{M_{\text{P}}^2}{2} \sqrt{\gamma} N [\gamma^{ij} \gamma^{kd} (K_{ik} K_{jd} - K_{ij} K_{kd}) + R] - \frac{1}{2} m^2 M_{\text{P}}^2 \sqrt{\gamma} N F_1([\mathfrak{R}], [\mathfrak{R}^2], [\mathfrak{R}^3]) - \frac{1}{2} m^2 M_{\text{P}}^2 \sqrt{\tilde{\gamma}} M F_2([\mathcal{K}], [\mathcal{K}^2], [\mathcal{K}^3]) \\ & + \frac{m^4 M_{\text{P}}^2 \lambda^2 M^2}{64N} \sqrt{\tilde{\gamma}} \gamma_{ik} \gamma_{jd} (2\Theta^{ij} \Theta^{kd} - \Theta^{ik} \Theta^{jd}) + \lambda \sqrt{\tilde{\gamma}} \left[C_\zeta - \frac{1}{4} m^2 M_{\text{P}}^2 M K_{ij} \Theta^{ij} \right] + \sqrt{\tilde{\gamma}} (D_j \lambda^i) C^j_i. \end{aligned} \quad (24)$$

It should be pointed out that the constraints imposed, at the level of the Lagrangian, impose a nontrivial relation not only on the three-dimensional metric, but also on the extrinsic curvature. This structure then is intrinsically different from the Lorentz-breaking massive gravity theories of [38,39].

The idea behind MTMG/eMTMG is to have a viable theory of massive gravity. Still this motivation led us extending MTMG into eMTMG. Part of this process consists of removing unwanted extra modes. Then the theory here introduced, in the gravity sector, is left only with the two transverse-traceless gravitational waves, hence we call it as ‘‘minimal theory.’’

It is interesting to notice that this theory is different from other widely known modified theories of gravity like scalar-tensor and vector-tensor theories, which introduce in general additional degrees of freedom.

The bottom line here is that we have extended MTMG to a more general class of massive gravity theories, which all only possess, at the fully nonlinear level, two tensor-type degrees of freedom on any background. We call the new theory the eMTMG.³ The original MTMG is a particular case of eMTMG and it can be refound when the functions $F_{1,2}$ reduce to this special form:

$$\begin{aligned} F_1^{\text{MTMG}} = & c_1 \left(\frac{1}{3} [\mathfrak{R}^3] - \frac{1}{2} [\mathfrak{R}] [\mathfrak{R}^2] + \frac{1}{6} [\mathfrak{R}]^3 \right) \\ & + \frac{1}{2} c_2 ([\mathfrak{R}]^2 - [\mathfrak{R}^2]) + c_3 [\mathfrak{R}] + c_4, \end{aligned} \quad (25)$$

$$\begin{aligned} F_2^{\text{MTMG}} = & c_1 [\mathcal{K}] + \frac{1}{2} c_2 ([\mathcal{K}]^2 - [\mathcal{K}^2]) \\ & + c_3 \left(\frac{1}{3} [\mathcal{K}^3] - \frac{1}{2} [\mathcal{K}] [\mathcal{K}^2] + \frac{1}{6} [\mathcal{K}]^3 \right), \end{aligned} \quad (26)$$

as shown in Appendix B.

³We name these models as ‘‘extended’’ because they possess a general graviton mass term at the level of the Hamiltonian.

III. HOMOGENEOUS AND ISOTROPIC BACKGROUND

So far we have extended the original MTMG theory to a much larger class of theories which is defined out of two free functions $F_{1,2}$ each dependent on three variables. This class of theories is expected to include a very large set of possibilities in terms of phenomenology. However, the original motivation to introduce such a class of theories was, and still is, to cure the problems encountered in the normal branch of MTMG, namely the presence of a pole in the function G_{eff}/G_N , which would in turn lead to an unviable cosmology in a neighborhood of them.⁴ Then it would be interesting to study whether inside the class of eMTMG theories, it is possible to find a subset which is always phenomenologically acceptable. By ‘‘always’’ we mean for any redshift and for any background dynamics. This extra dynamical condition might be too strong, as effectively, one would need only a subset of well-defined dynamics, however, after imposing it, if such a subset existed, would provide a ghost-free, instability-free arena, where we can try to solve today’s tensions in cosmology out of a massive graviton.

Hence, let us explore these extended models as to find a good behavior for G_{eff}/G_N , the effective gravitational constant for the density perturbations of a pressureless fluid on a homogeneous and isotropic background. For this aim, let us study in this section, first of all, the background for these theories in the presence of matter fields. Let us focus then on a spatially flat FLRW background which is described by

$$\begin{aligned} N = N(t), \quad N^i = 0, \quad \gamma_{ij} = a(t)^2 \delta_{ij}, \\ \lambda = \lambda(t), \quad \lambda^i = 0, \end{aligned} \quad (27)$$

whereas the fiducial sector is given by

$$M = M(t), \quad \tilde{\gamma}_{ij} = \tilde{a}(t)^2 \delta_{ij}. \quad (28)$$

For the matter sector we introduce a perfect fluid (one for each matter component) modeled by the Schutz-Sorkin action as in [40–42]

⁴At the pole or approaching the pole, at least at linear order, the theory would exit the regime of validity of a low-energy effective theory description.

$$S_m = - \int d^4x \sqrt{-g} [\rho(n) + J^\mu \partial_\mu \ell], \quad n \equiv \sqrt{-J^\mu J^\nu g_{\mu\nu}}, \quad (29)$$

for which we can introduce the normalized fluid 4-velocity as $u^\alpha = J^\alpha/n$, and $g_{\mu\nu}$ is the four-dimensional physical metric written in the ADM splitting (2). On a spatially flat FLRW background we have at the level of the background

$$J^0(t) = \frac{\mathcal{J}(t)}{N(t)}, \quad \mathcal{J}(t) = n(t) = \frac{\mathcal{N}_0}{a^3}, \quad (30)$$

and the proportionality constant \mathcal{N}_0 determines the constant number of fluid particles (n being their number density). Furthermore, the background equations of motion imply

$$\ell(t) = - \int^t N(t') \rho_{,n}(t') dt'. \quad (31)$$

For the spatially flat FLRW background we find

$$[\mathfrak{K}^n] = 3 \left(\frac{\tilde{a}}{a} \right)^n, \quad [\mathcal{K}^n] = 3 \left(\frac{a}{\tilde{a}} \right)^n, \quad (32)$$

so that the minisuperspace⁵ Lagrangian density evaluated on the background reduces to

$$\begin{aligned} \mathcal{L}_{\text{mini}} = & \frac{3}{2} \left[\dot{a} (3F_{2, [\mathcal{K}^3]} a^2 + 2F_{2, [\mathcal{K}^2]} a \tilde{a} + F_{2, [\mathcal{K}]} \tilde{a}^2) \frac{M}{N} - \dot{\tilde{a}} (a^2 F_{1, [\mathfrak{K}]} + 2F_{1, [\mathfrak{K}^2]} a \tilde{a} + 3F_{1, [\mathfrak{K}^3]} \tilde{a}^2) \right] m^2 M_{\text{P}}^2 \lambda - \frac{3M_{\text{P}}^2 a \dot{a}^2}{N} \\ & - \frac{3M^2 (3F_{2, [\mathcal{K}^3]} a^2 + 2F_{2, [\mathcal{K}^2]} a \tilde{a} + F_{2, [\mathcal{K}]} \tilde{a}^2)^2 m^4 M_{\text{P}}^2 \lambda^2}{16aN} - \frac{m^2 M_{\text{P}}^2 \tilde{a}^3 (M F_2 + N F_1)}{2} \\ & - a^3 \sum_I [\mathcal{J}_I \dot{\ell}_I + N \rho_I(\mathcal{J}_I)]. \end{aligned} \quad (33)$$

On evaluating the Euler-Lagrange equations for the fields N , a , λ , ℓ_I and \mathcal{J}_I we find the equations of motion for the background.

In order to evaluate the value of λ on the background it turns out to be much simpler to study the Hamilton equations of motion. Out of the Lagrangian we can find the Hamiltonian in the minisuperspace via a Legendre transformation as

$$\begin{aligned} H_{\text{mini}} = & \left(\frac{p_a (3F_{2, [\mathcal{K}^3]} a^2 + 2F_{2, [\mathcal{K}^2]} a \tilde{a} + F_{2, [\mathcal{K}]} \tilde{a}^2) M}{4a} + \frac{3M_{\text{P}}^2 \dot{\tilde{a}} (a^2 F_{1, [\mathfrak{K}]} + 2F_{1, [\mathfrak{K}^2]} a \tilde{a} + 3F_{1, [\mathfrak{K}^3]} \tilde{a}^2)}{2} \right) m^2 \lambda \\ & + \frac{m^2 M_{\text{P}}^2 \tilde{a}^3 M F_2}{2} + N \left(\frac{F_1 a^3 m^2 M_{\text{P}}^2}{2} + a^3 \sum_I \rho_I(\mathcal{J}_I) - \frac{p_a^2}{12a M_{\text{P}}^2} \right) + \sum_I l_I p_{\mathcal{J}_I} + \sum_I \tilde{l}_I (\mathcal{J}_I a^3 + p_{\ell_I}), \end{aligned} \quad (34)$$

where p_a , $p_{\mathcal{J}_I}$ and p_{ℓ_I} are the momenta conjugate to the variables a , \mathcal{J}_I and ℓ_I respectively, whereas λ , N , l_I and \tilde{l}_I are all Lagrange multipliers which set constraints. One such constraint is then

$$C_0 = \frac{p_a (3F_{2, [\mathcal{K}^3]} a^2 + 2F_{2, [\mathcal{K}^2]} a \tilde{a} + F_{2, [\mathcal{K}]} \tilde{a}^2) M}{4a} + \frac{3M_{\text{P}}^2 \dot{\tilde{a}} (a^2 F_{1, [\mathfrak{K}]} + 2F_{1, [\mathfrak{K}^2]} a \tilde{a} + 3F_{1, [\mathfrak{K}^3]} \tilde{a}^2)}{2} \approx 0, \quad (35)$$

whereas the Hamiltonian constraint can be written as

$$R_0 = \frac{F_1 a^3 m^2 M_{\text{P}}^2}{2} + a^3 \sum_I \rho_I(\mathcal{J}_I) - \frac{p_a^2}{12a M_{\text{P}}^2} \approx 0. \quad (36)$$

Let us now impose that the time derivative of the constraints should vanish on the constraint surface. For example we have

$$\dot{p}_{\mathcal{J}_I} = \{p_{\mathcal{J}_I}, H_{\text{mini}}\} = -a^3 (N \rho_{I, \mathcal{J}} + \tilde{l}_I) \approx 0, \quad \text{or} \quad \tilde{l}_I \approx -N \rho_{I, \mathcal{J}}, \quad (37)$$

which sets all the \tilde{l}_I 's Lagrange multipliers in the matter sectors. Furthermore we have

⁵We have evaluated the Lagrangian density (33) by using only the fields which can fully describe a homogeneous and isotropic manifold. For example, in the metric tensor, this approach only keeps then the lapse $N(t)$ and the scale factor $a(t)$. In general we have several fields living in one-dimensional space that we call minisuperspace.

$$\{\mathcal{J}_I a^3 + p_{\ell I}, H_{\text{mini}}\} = 3\mathcal{J}_I a^2 \left(\frac{Mm^2(3F_{2,[\mathcal{K}^3]}a^2 + 2F_{2,[\mathcal{K}^2]}a\tilde{a} + F_{2,[\mathcal{K}]} \tilde{a}^2)\lambda}{4a} - \frac{p_a N}{6M_{\text{P}}^2 a} \right) + a^3 l_I \approx 0, \quad (38)$$

which can be used in order to set the Lagrange multipliers l_I 's. Also we can find

$$\dot{R}_0 \equiv \{R_0, H_{\text{mini}}\} + \frac{\partial R_0}{\partial \tilde{a}} \dot{\tilde{a}} \approx 0, \quad (39)$$

which combined with C_0 , gives

$$\dot{R}_0 - C_0 \approx f\lambda \approx 0, \quad (40)$$

where f is a quantity which in general does not vanish, unless some fine-tuned dynamics are considered. This equation then determines the Lagrange multiplier λ , without adding any new constraint, and it finally leads to the conclusion that on the constraint surface, that is on the background, we need to impose $\lambda(t) = 0$.

With $\lambda(t) = 0$, the independent background equations of motion greatly simplify and reduce to

$$3M_{\text{P}}^2 H^2 = \sum_I \rho_I + \frac{1}{2} M_{\text{P}}^2 m^2 F_1, \quad (41)$$

$$H \frac{M}{N} \left[\left(\frac{\tilde{a}}{a} \right)^2 F_{2,[\mathcal{K}]} + 2 \left(\frac{\tilde{a}}{a} \right) F_{2,[\mathcal{K}^2]} + 3F_{2,[\mathcal{K}^3]} \right] = \frac{\dot{\tilde{a}}}{N\tilde{a}} \frac{\tilde{a}}{a} \left[F_{1,[\mathfrak{S}]} + 2 \left(\frac{\tilde{a}}{a} \right) F_{1,[\mathfrak{S}^2]} + 3 \left(\frac{\tilde{a}}{a} \right)^2 F_{1,[\mathfrak{S}^3]} \right], \quad (42)$$

$$\frac{\dot{\rho}_I}{N} = -3H(\rho_I + P_I), \quad (43)$$

where $H \equiv \dot{a}/(Na)$ is the Hubble expansion rate for the physical metric. Let us then define

$$X \equiv \frac{\tilde{a}}{a}, \quad (44)$$

and suppose that $X > 0$, during the whole evolution of the universe in the regime of interest. The constraint equation, Eq. (42), can be rewritten as

$$H \frac{M}{N} (X^2 F_{2,[\mathcal{K}]} + 2XF_{2,[\mathcal{K}^2]} + 3F_{2,[\mathcal{K}^3]}) = \left(\frac{\dot{X}}{N} + HX \right) (F_{1,[\mathfrak{S}]} + 2XF_{1,[\mathfrak{S}^2]} + 3X^2 F_{1,[\mathfrak{S}^3]}). \quad (45)$$

Unlike the original MTMG, the space of solutions for Eq. (45) is not in general separated into two branches, the so called self-accelerating and normal branches. The special case in which such separation occurs is briefly studied in Appendix C. On the other hand, in the rest of the present paper we consider the general case, that is the single universal branch, defined by Eq. (45), for all the eMTMG models.

IV. LINEAR PERTURBATIONS

In this section we study linear perturbations around the spatially flat FLRW background introduced in the previous section.

A. Gravitational waves

Let us now consider the tensor perturbations for the physical metric, namely

$$N = N(t), \quad N^i = 0, \quad \gamma_{ij} = a^2 \left[\delta_{ij} + \sum_{\lambda=+,\times} \epsilon_{ij}^\lambda h_\lambda \right], \quad (46)$$

where the two symmetric polarization tensors satisfy both the transverse and traceless conditions $\epsilon_{ij}^\lambda \delta^{ij} \partial_l h_\lambda = 0$, $\delta^{ij} \epsilon_{ij}^\lambda = 0$ and the chosen normalizations $\epsilon_{ij}^+ \epsilon_{lm}^+ \delta^{il} \delta^{jm} = 1 = \epsilon_{ij}^\times \epsilon_{lm}^\times \delta^{il} \delta^{jm}$, together with $\epsilon_{ij}^+ \epsilon_{lm}^\times \delta^{il} \delta^{jm} = 0$. After expanding the Lagrangian at the second order in the tensor perturbations, we obtain the quadratic action describing their dynamics as

$$S = \frac{M_{\text{P}}^2}{8} \sum_{\lambda=+,\times} \int d^4x N a^3 \left[\left(\frac{\dot{h}_\lambda}{N} \right)^2 - \frac{(\partial h_\lambda)^2}{a^2} - \mu^2 h_\lambda^2 \right], \quad (47)$$

where

$$\begin{aligned} \mu^2 = & \frac{1}{2}m^2X[r(X^2F_{2,[\mathcal{K}]} + 4XF_{2,[\mathcal{K}^2]} + 9F_{2,[\mathcal{K}^3]}) \\ & + F_{1,[\mathfrak{R}]} + 4XF_{1,[\mathfrak{R}^2]} + 9X^2F_{1,[\mathfrak{R}^3]}], \end{aligned} \quad (48)$$

and we have defined for later convenience also the quantity

$$r \equiv \frac{1}{X} \frac{M}{N}. \quad (49)$$

Therefore these models do introduce a nontrivial mass for the tensor modes, however the speed of propagation, for high- k modes, i.e. at energies for which the graviton becomes ultrarelativistic, will still be equal to unity. Furthermore, the graviton mass not only does not vanish in general, but also it is changing with time. For this reason, we will also demand that a well-behaved subset of eMTMG models would also satisfy the condition of a non-negative μ^2 for any dynamics of the background.

B. Effective gravitational constant

In this section we consider instead the scalar perturbations and expand the action for the theory in the presence of matter perfect fluids up to the second order and remove at the level of the action all the auxiliary fields to find the quadratic action for the field $\delta\rho/\rho$, which we shall define below.

First of all we will explicitly write down all the perturbation variables, both in the gravity and in the matter sector. We introduce scalar perturbations for the physical metric in the following way:

$$N = N(t)(1 + \alpha), \quad (50)$$

$$N_i = N(t)\partial_i\chi, \quad (51)$$

$$\gamma_{ij} = a(t)^2\delta_{ij}(1 + 2\zeta) + 2\partial_{ij}E. \quad (52)$$

Since we have fixed from the beginning the unitary gauge, we cannot impose any gauge condition on the perturbation variables. We also need to introduce perturbations for the following eMTMG variables

$$\lambda = \delta\lambda, \quad \lambda^i = \frac{1}{a^2}\delta^{ij}\partial_j\delta\lambda_V. \quad (53)$$

As for the matter sectors we proceed instead as follows. First of all we make the following split

$$J^0 = \frac{\mathcal{J}(t)}{N(t)}(1 + \delta J), \quad (54)$$

$$J^i = \frac{1}{a^2}\delta^{ij}\partial_j\delta J_V, \quad (55)$$

$$\ell = \ell(t) + \delta\ell. \quad (56)$$

For each matter component we consider matter field redefinitions as follows. We first define the fluid perturbation scalar velocity v as

$$u_i = g_{i\mu}u^\mu = \partial_iv, \quad (57)$$

which leads to the field redefinition

$$\delta J_V = n(t)(v - \chi). \quad (58)$$

Expanding the action at the second order in the perturbation variables, finding the equation of motion for v and solving it for $\delta\ell$ gives

$$\delta\ell = \rho_{,n}v, \quad (59)$$

which can be used in order to integrate out the field $\delta\ell$. Also we can perform a field redefinition as follows

$$\delta J = \frac{\rho}{n\rho_{,n}} \frac{\delta\rho}{\rho} - \alpha, \quad \text{where } \frac{\delta\rho}{\rho} \equiv \frac{\rho}{\rho(t)} - 1. \quad (60)$$

As for now we have an action for the perturbation which is a function of the following variables: α , χ , ζ , E , and $\delta\lambda$, $\delta\lambda_V$ in the metric sector, together with $\delta\rho/\rho$ and v for each matter-fluid component. We can find equations of motion for each of these perturbation variables, and we label them, e.g. as E_χ (which vanish, i.e. $E_\chi = 0$, and the subscript shows the variables for which the equation of motion is derived, χ in this example). In the following, although not necessary, we will use time reparametrization as to set $N(t) = a(t)$. Since we want to match the phenomenology with observations we will also make the following field redefinitions which link α and ζ to the gauge invariant definitions of the Bardeen potentials ψ and ϕ :

$$\alpha = \psi - \frac{1}{a}\dot{\chi} + \frac{1}{a}\frac{d}{dt}\left[a\frac{d}{dt}\left(\frac{E}{a^2}\right)\right], \quad (61)$$

$$\zeta = -\phi - H\chi + aH\frac{d}{dt}\left(\frac{E}{a^2}\right), \quad (62)$$

$$\frac{\delta\rho}{\rho} = \delta - \frac{\dot{\rho}}{a\rho}\chi + \frac{\dot{\rho}}{\rho}\frac{d}{dt}\left(\frac{E}{a^2}\right), \quad (63)$$

whereas the last equation introduces δ as the gauge invariant longitudinal matter perturbation. Finally we also make the field redefinition

$$v = -\frac{a}{k^2}\theta + \chi - a\frac{d}{dt}\left(\frac{E}{a^2}\right), \quad (64)$$

where θ is another gauge invariant variable related to the scalar fluid velocity. We can also introduce, at the level of

perturbation, a shear term for each matter component as done in [16].

So far, the equation of state of the perfect fluid is general and in the next subsection we shall further consider equations of motion for this general system. In the rest of this subsection, on the other hand, we restrict our considerations to the case of a single perfect fluid to compute its sound speed and in the case of dust, the effective gravitational constant.

The first nontrivial feature of the models consists in the constraint equation set by the field $\delta\lambda_\nu$. In fact we find that it can be written as

$$\zeta \propto \Xi_1 \frac{k^2}{a^2} E. \quad (65)$$

$$\begin{aligned} \Xi_1 \equiv & F_{2,[\mathcal{K}][\mathcal{K}]} + \frac{4}{X} F_{2,[\mathcal{K}][\mathcal{K}^2]} + \frac{6}{X^2} F_{2,[\mathcal{K}][\mathcal{K}^3]} + \frac{4}{X^2} F_{2,[\mathcal{K}^2][\mathcal{K}^2]} \\ & + \frac{12}{X^3} F_{2,[\mathcal{K}^2][\mathcal{K}^3]} + \frac{9}{X^4} F_{2,[\mathcal{K}^3][\mathcal{K}^3]} + 2F_{2,[\mathcal{K}^2]} + \frac{6}{X} F_{2,[\mathcal{K}^3]}. \end{aligned} \quad (66)$$

Therefore, the quantity Ξ_1 discriminates the behavior of the theory in the high- k regime, as, in general, the phenomenology of the theory will be different for the eMTMG models depending on whether Ξ_1 is zero (or negligible) or not. The mirror quantity Ξ_2 for the function F_1 turns out to have also a strong influence on the phenomenology of the theory, as we will see later on

$$\begin{aligned} \Xi_2 = & F_{1,[\mathfrak{R}][\mathfrak{R}]} + 4XF_{1,[\mathfrak{R}][\mathfrak{R}^2]} + 6X^2F_{1,[\mathfrak{R}][\mathfrak{R}^3]} \\ & + 4X^2F_{1,[\mathfrak{R}^2][\mathfrak{R}^2]} + 12X^3F_{1,[\mathfrak{R}^2][\mathfrak{R}^3]} \\ & + 9X^4F_{1,[\mathfrak{R}^3][\mathfrak{R}^3]} + 2F_{1,[\mathfrak{R}^2]} + 6XF_{1,[\mathfrak{R}^3]}. \end{aligned} \quad (67)$$

Indeed, one can proceed to remove all the auxiliary fields except for the field $\delta\rho/\rho$, which, in the case of a single fluid, has the following schematic quadratic Lagrangian density

$$\mathcal{L}_{\delta\rho} = A(k^2, t) \left[\frac{1}{N} \frac{\partial}{\partial t} \left(\frac{\delta\rho}{\rho} \right) \right]^2 + B(k^2, t) \left(\frac{\delta\rho}{\rho} \right)^2. \quad (68)$$

In the high- k regimes, we find that the no-ghost condition is always verified, since

$$A = \frac{1}{2} Na^3 \frac{a^2}{k^2} \frac{\rho^2}{n\rho_n} + \mathcal{O}(a^4/k^4), \quad (69)$$

which is always positive, provided that $n\rho_n = \rho + P > 0$. As for the B term, we need to distinguish among possibilities.

(i) Case for which $\Xi_1 \neq 0$, and in this case we have

$$B = -\frac{Na^3}{2} \frac{\rho_{,nn}\rho^2}{\rho_{,n}^2} + \mathcal{O}(a^2/k^2), \quad \text{or} \quad c_s^2 = \frac{n\rho_{,nn}}{\rho_{,n}}, \quad (70)$$

giving the standard results for the propagation of perturbations in a fluid.

(ii) Case for which $\Xi_1 = 0$, or very negligible namely $\Xi_1 k^2/(a^2 H^2) \ll 1$, and in this case we find instead

$$\begin{aligned} B = & -\frac{Na^3}{2} \left[\frac{\rho_{,nn}\rho^2}{\rho_{,n}^2} + \Xi_2 B_2(t) \frac{m^2 \rho^2}{M_{\text{P}}^2 H^4} \right] + \mathcal{O}(a^2/k^2), \\ \text{or} \quad c_s^2 = & \frac{n\rho_{,nn}}{\rho_{,n}} + \Xi_2 B_2 \frac{m^2 \rho + P}{H^2 M_{\text{P}}^2 H^2}, \end{aligned} \quad (71)$$

which leads to a nontrivial propagation speed unless also $\Xi_2 = 0$ (or, as mentioned above, very negligible). Indeed the case $\Xi_1 = 0 = \Xi_2$ is the one we are going to focus on in the following sections.⁶ Nonetheless, as long as Ξ_2 (or B_2) does not vanish, the speed of propagation for any matter fluid will get modified. This nontrivial property is shared by another minimal theory of gravity introduced and studied in [43,44]. In particular, a pressureless fluid will acquire a nontrivial contribution. This unusual behavior of matter fields can strongly constrain the phenomenology of the theories belonging to this special case.

(iii) Case for which $\Xi_1 = 0 = \Xi_2$ and $\rho_{,nn} = 0 = c_s^2$, where the last equation of state corresponds to choosing a pressureless fluid as matter field. In this case the Lagrangian density of Eq. (68) for the energy-density perturbations reduces to

$$\begin{aligned} \mathcal{L}_{\text{dust}} = & \frac{1}{2} Na^3 \frac{a^2}{k^2} \rho \left\{ \left[\frac{1}{N} \frac{\partial}{\partial t} \left(\frac{\delta\rho}{\rho} \right) \right]^2 \right. \\ & \left. + 4\pi G_{\text{eff}} \rho \left(\frac{\delta\rho}{\rho} \right)^2 \right\}, \end{aligned} \quad (72)$$

out of which one can deduce the expression for G_{eff}/G_N , whose value (which is not unity, in general) is explicitly written in Appendix D.

C. Equations of motion for scalar perturbations

In this subsection, instead of considering a Lagrangian approach, we consider an equivalent approach, based on studying the equations of motion for scalar perturbations in the presence of matter fields modeled by perfect fluids with

⁶Since $B_2 \propto (X^2 F_{2,[\mathcal{K}]} + 2XF_{2,[\mathcal{K}^2]} + 3F_{2,[\mathcal{K}^3]})^2 (F_{1,[\mathfrak{R}]} + 2XF_{1,[\mathfrak{R}^2]} + 3X^2 F_{1,[\mathfrak{R}^3]})^2$, the case $B_2 = 0$ corresponds to the self-accelerating case, which will be discussed in Appendix C.

general equations of state. In this case though, we generalize the previous results to the case of an arbitrary number of matter fields.

In particular, in terms of the gauge invariant variables introduced in Eqs. (61)–(64), the matter equations of motion for each matter component are the same as in General Relativity, namely

$$\dot{\delta}_I = 3a(w_I - c_{sI}^2)H\delta_I - (1 + w_I)(\theta_I - 3\dot{\phi}), \quad (73)$$

$$\dot{\theta}_I = aH(3c_{sI}^2 - 1)\theta_I + k^2\psi + \frac{c_{sI}^2 k^2}{1 + w_I}\delta_I - k^2\sigma_I, \quad (74)$$

where $w_I \equiv P_I/\rho_I$, and $c_{sI}^2 \equiv \dot{p}_I/\dot{\rho}_I = (\frac{\partial p_I}{\partial \rho_I})_s$ is the speed of propagation for each matter species. Here the subscript I runs over all the standard matter components we consider. The fact that in the matter sector we refined the same equations of GR is not surprising, as the Lagrangians of matter fields do satisfy general covariance.

We can proceed by solving $E_{\delta\lambda_V}$ for χ , E_α for E and E_χ for $\delta\lambda$. Now the equation of motion E_E can be written as

$$E_E = S_1\dot{\phi} + S_2\phi + S_3\psi + \sum_I S_{4,I}\dot{\delta}_I + \sum_I S_{6,I}\delta_I + \sum_I S_{8,I}\dot{\theta}_I + \sum_I S_{10,I}\theta_I + S_{12}\delta\lambda_V = 0, \quad (75)$$

where the S 's coefficients are functions of k and time.⁷ A linear combination of E_E and E_ζ leads instead to

$$E_{E\zeta} = T_1\delta\lambda_V + T_2\phi + T_3\psi + \sum_I T_{4,I}\delta_I + \sum_I T_{6,I}\theta_I + \sum_I T_{6,I}\sigma_I = 0, \quad (76)$$

which can be used to define $\delta\lambda_V$ in terms of the other variables. Finally the equation $E_{\delta\lambda}$ leads to

$$E_{\delta\lambda} = U_1\phi + \sum_I U_{2,I}\delta_I + \sum_I U_{4,I}\theta_I = 0. \quad (77)$$

Here, T 's and U 's are, once more, coefficients which depend on k and time t . On considering the time derivative of Eq. (77), namely $\dot{E}_{\delta\lambda}$, and replacing $\dot{\phi}$, $\dot{\delta}_I$ and $\dot{\theta}_I$ with those given by Eqs. (75), (73), and (74) respectively, we arrive at the so-called “shear equation” for this theory. This approach then leads to a structure of the equations of motion which is analogue to the standard approach in General Relativity. This is a consequence of the fact that these theories do not add any new degree of freedom, i.e. no new dynamical equation is necessary to determine new

fields (which do not exist in the first place), but still they change the equations of motion for the linear perturbation, e.g. by the fact that the coefficients $S_{1,\dots,12}$ differ from the ones in GR. So after all, a different phenomenology is expected to take place in general, although the number of degrees of freedom has not been changed.

V. PHENOMENOLOGICAL CRITERIA

So far the approach was totally general, and it can be applied to general functions $F_{1,2}$ and arbitrary matter components. On the other hand, in order to give predictions on the graviton mass from observational data, it is ideal to have a well-motivated subclass of models with a finite number of parameters. For this reason, in this section we shall make a list of phenomenologically motivated criteria to be imposed on the theory.

A. $c_s^2 = 0$ at all times and $G_{\text{eff}}/G_N \rightarrow 1$ at early times for pressureless fluids

In Sec. IV B we have computed the squared sound speed c_s^2 and the effective gravitational constant G_{eff} for scalar perturbations in the presence of a pressureless fluid. Based on these considerations and the result of Sec. IV B, we can divide the theories into the following three categories.

- (1) The case $\Xi_1 \neq 0$. In this case, from the equation of motion for λ^i , we find $\zeta \propto E$. Furthermore, after removing all the auxiliary fields, we find that $G_{\text{eff}}/G_N = 1 + \mathcal{O}([k/(aH)]^{-2})$ independently of the model. Therefore, for this class of models, the background dynamics will be modified, but the behavior of the perturbations, in the short scales regime, will not. This could be an interesting possibility, but since, we are looking for theories which can address a gravitational interaction weaker than General Relativity, we will not discuss this model further in this paper, but we will consider it as a subject of investigation for a future project.
- (2) The case $\Xi_1 = 0$, but $\Xi_2 \neq 0$. In this case, $\zeta = 0$, and the matter perturbations acquire a nonzero speed of propagation, namely $c_s^2 \neq 0$. This case will not be discussed further as strongly constrained from a phenomenological point of view.
- (3) The case $\Xi_1 = 0 = \Xi_2$. In this case, $\zeta = 0$, $G_{\text{eff}}/G_N = A(t)/Z(t) + \mathcal{O}([k/(aH)]^{-2})$, where $A(t)$ and $Z(t)$ are expressions which only depend on time. In general $G_{\text{eff}}/G_N \neq 1$, but

$$\lim_{m/H \rightarrow 0} \frac{G_{\text{eff}}}{G_N} = 1, \quad (78)$$

which leads to the standard evolution for matter perturbations at early times.

Both c_s^2 and G_{eff} are important for the formation of the large-scale structure in the universe. In GR, c_s^2 for a

⁷The explicit form of the coefficients $S_{1,\dots,12}$ are in general quite lengthy, and not strongly illuminating. We will instead write the explicit expressions of the observables when needed.

pressureless fluid vanishes and this is consistent with observations. If c_s^2 is negative or too large then the prediction of the theory would contradict with observations. Although a positive and sufficiently small c_s^2 can in principle be consistent, for simplicity we restrict our considerations to the case with $c_s^2 = 0$ for the eMTMG. In this case the effective gravitational constant controls the behavior of scalar perturbations. Motivated by the fact that the standard Λ CDM in GR fits the Planck data very well, we demand that $G_{\text{eff}}/G_N \rightarrow 1$ at early times. On the other hand, at late times we would like to have deviation of G_{eff}/G_N from unity. This is motivated by the recent tension in the estimation of the parameter S_8 between the Planck 2018 data and late time observations like KiDS and DES. Hence, modifying the growth of structure formation may address the S_8 tension in cosmology. For these reasons, in the rest of the present paper we study the third case above, that is the case for which

$$\Xi_1 = 0 = \Xi_2. \quad (79)$$

B. Finite G_{eff}/G_N for $\forall X(t)$

One of the main motivations in this paper is to have models for which G_{eff}/G_N never blows up during a general evolution of $X(t)$ and $r(t)$, keeping the theory safely in the regime of validity of the effective field theory. This implies that, on considering the quantity $Z(t)$, we need to impose that it never vanishes for any dynamics of $X(t)$ and $r(t)$, assuming them to be all positive quantities.

If the expression for G_{eff}/G_N has poles for some real value of $X = X_\infty$, we could just make sure that during the dynamics these values of X_∞ should not be reached. However, this might not be possible to predict in general. For example in MTMG, this pole corresponds to a time for which $H^2 = H_\infty^2 = \mu^2/2$, where μ is the mass of the graviton. This means that for MTMG we had to have that $H_\infty^2 < H_0^2$. However, well before reaching this pole, the phenomenology of the theory was leading to inconsistencies, see e.g. [37,45]. Therefore, just avoiding the poles may not be enough to lead to a viable phenomenology. Therefore, we want to find, if possible, a subset of theories which, under this point of view, are always consistent.

C. Positive μ^2 for $\forall X(t)$

Furthermore, we also impose that the squared mass μ^2 of the tensor modes remains finite and positive during a general evolution of $X(t)$ and $r(t)$.

As previously mentioned, in MTMG, a pole for G_{eff}/G_N was reached when $H^2 = H_\infty^2 = \mu^2/2$. Evidently this pole can be removed provided that we impose $\mu^2 < 0$, that is when the graviton has a negative mass squared. What would this mean? In a cosmological scenario, this would make tensor modes unstable. But this instability

would be reached when the energy of the graviton itself, $E = \sqrt{k_{\text{phys}}^2 - |\mu^2|}$ is comparable to $\sqrt{|\mu^2|}$, and as such astrophysically produced gravitons will not show this instability. Furthermore, such an instability would have a typical time-of-instability of order of $1/\sqrt{|\mu^2|} \simeq H_0^{-1}$, which would become evident only in the future.

Nonetheless, in this paper we assume the tensor modes to be nontachyonic. In this case we have to impose that during the whole history of the universe $\mu^2 \geq 0$. This condition is, in general, independent of the absence of poles in G_{eff}/G_N , so we expect that they can be imposed simultaneously.

D. Finite ISW effect

There are other observables which still strongly influence the behavior of late time cosmology. In particular, here we consider the following combination, whose time derivative affects the ISW effect, namely

$$\psi_{\text{ISW}} \equiv \phi + \psi. \quad (80)$$

We demand that ψ_{ISW} remains finite during a general evolution of $X(t)$ and $r(t)$.

In fact, the correlation between ISW and galaxy perturbations usually sets strong constraints for modified gravity models, in particular an anticorrelation signal is ruled out. For example in MTMG, it was shown that the ISW-galaxy correlation effect was one of the tightest bound the theory had to pass, see e.g. [24,36,45,46]. As already stated above, in order not to have strong constraints coming from this observable, we demand the finiteness of ψ_{ISW} and its time derivative during the whole dynamics of the universe. Doing so in principle should add new constraints, and as such further reduce the possibilities for the model to exist. However, as we shall see later on, at least for the case under study the finiteness of G_{eff}/G_N turns out to be a sufficient condition for the finiteness of the ISW observable.

VI. CONCRETE REALIZATION BASED ON POLYNOMIAL ANSATZ FOR $F_{1,2}$

In principle it should be possible to find the subclass consisting of all theories that satisfy the phenomenological criteria summarized in the previous section. However, the analysis and the result are expected to be rather complicated (if possible in practice). In this section we therefore consider a simple ansatz for the functions $F_{1,2}$ and then impose the phenomenological criteria step by step.

Considering the fact that the original MTMG has polynomial expressions for $F_{1,2}$ in terms of their variables, we therefore restrict our considerations to the case where $F_{1,2}$ are polynomials of their arguments. Furthermore, for simplicity we truncate the polynomials at the sixth order in \mathcal{K}_j^i and \mathcal{R}_j^i , respectively, as

$$\begin{aligned}
F_1(A, B, C) = & a_{3,111111}A^6 + a_{3,11111}A^5 + a_{3,1111}A^4 + a_{3,111}A^3 + a_{3,112}A^2B + a_{3,222}B^3 + a_{2,11}A^2 + 2a_{2,12}AB \\
& + a_{3,122}AB^2 + 2a_{2,13}AC + a_{2,22}B^2 + 2a_{2,23}BC + a_{2,33}C^2 + a_{1,1}A + a_{1,2}B + a_{1,3}C \\
& + g_1A^4B + g_2A^2B^2 + g_3A^3B + g_4A^2C + g_5A^3C + g_6ABC + c_4,
\end{aligned} \tag{81}$$

$$\begin{aligned}
F_2(A, B, C) = & b_{3,111111}A^6 + b_{3,11111}A^5 + b_{3,1111}A^4 + b_{3,111}A^3 + b_{3,112}A^2B + b_{3,222}B^3 + b_{2,11}A^2 + 2b_{2,12}AB \\
& + b_{3,122}AB^2 + 2b_{2,13}AC + b_{2,22}B^2 + 2b_{2,23}BC + b_{2,33}C^2 + b_{1,1}A + b_{1,2}B + b_{1,3}C \\
& + h_1A^4B + h_2A^2B^2 + h_3A^3B + h_4A^2C + h_5A^3C + h_6ABC.
\end{aligned} \tag{82}$$

Essentially, as already stated above, the polynomials have been chosen so as to be linear combinations of terms in the form $A^{b_1}B^{b_2}C^{b_3}$, with the conditions that $b_i \in \mathbb{N}^+$ and $b_3 \leq 2$. Then the other powers, $b_{1,2}$, have been chosen so that, on FLRW, once written as polynomials in X , which can be grouped according to equal powers of X . For example, the variable C in F_1 will lead to a term X^3 , B to X^2 and A to X . Therefore, for instance, we allow terms in $C^2, A^3C, ABC, B^3, A^2B^2, A^4B, A^6$ which all lead to a term proportional to X^6 , etc. Although this toy model is just meant to be a proof-of-existence case, we will find that the models satisfying the properties we are looking for all behave in the same way on the FLRW background, so that we believe the model can catch general properties of the extended minimal models of gravity. Indeed, as we shall see later on, further simplified subcases which are still general enough in their dynamics will be found.

In the following we shall impose all phenomenological criteria considered in the previous section step by step.

A. $c_s^2 = 0$ at all times and $G_{\text{eff}}/G_N \rightarrow 1$ at early times

Let us now impose the conditions on the parameters so that $c_s^2 = 0$ at all times, that $G_{\text{eff}}/G_N \rightarrow 1$ at early times and yet that G_{eff}/G_N exhibits interesting deviation from unity at late times. As discussed in Sec. VA, this amounts to requiring that, for any X , we have $\Xi_1 = 0 = \Xi_2$. This will select the models belonging to the third case mentioned above. This condition fixes some constant parameters to satisfy the following relations

$$a_{3,111111} = -\frac{7}{135}a_{3,222} - \frac{1}{45}a_{2,33}, \tag{83}$$

$$a_{3,11111} = -\frac{2}{15}a_{2,23} - \frac{7}{45}a_{3,122}, \tag{84}$$

$$a_{3,1111} = -\frac{4}{9}a_{2,13} - \frac{5}{27}a_{2,22} - \frac{4}{9}a_{3,112}, \tag{85}$$

$$a_{3,111} = -\frac{1}{3}a_{1,3} - \frac{10}{9}a_{2,12}, \tag{86}$$

$$a_{2,11} = -a_{1,2}, \tag{87}$$

with analogue relations holding for the b 's coefficients.

B. Finite G_{eff}/G_N for $\forall X(t)$

We now require that $Z(t)$ never vanishes for any positive $X(t)$ and $r(t)$. Later we shall also impose the positivity of $\mu^2(t)$. Therefore, in this subsection we also assume that $\mu^2(t)$ is also positive.

In order to simplify the expression for $Z(t)$, we first replace \dot{X} on using the background constraint, Eq. (45). We also replace \dot{a} in terms of H , M in terms of r , N , X , and say $b_{3,222}$ in terms of μ^2 , by inverting Eq. (48). Then we find that $Z \propto Z_1(t)^2 Z_2(t)^2 Z_3(t)^2 Z_4(t)^2$ (where the proportionality factor is positive definite, by assumptions, being a product of powers of H , r , X , a , N) and Z_I ($I = \{1, \dots, 4\}$) are instead polynomial in powers of X , r and μ^2 . We conservatively impose that each of the coefficients of such polynomials have the same sign, so that each polynomial Z_I would never vanish. The expressions $Z_{3,4}$ only set constraints on the a 's parameters, whereas $Z_{1,2}$ also constrain the b 's parameters. On considering only $Z_{2,3,4}$ (Z_1 being the most complicated expression) then we find the following constraints need to be satisfied

$$a_{1,1} = -A_{1,1}^2, \quad A_{1,1}^2 \geq 0, \tag{88}$$

$$a_{1,2} = A_{1,2}^2 \geq 0, \tag{89}$$

$$a_{2,12} = -\frac{1}{2}a_{1,3} + \xi^2, \quad \xi^2 \geq 0, \tag{90}$$

$$a_{2,22} = -3a_{2,13} - \frac{3}{2}a_{3,112}, \tag{91}$$

$$a_{2,23} = -\frac{3}{4}a_{3,122} - \frac{9}{8}(g_3 + g_4), \tag{92}$$

$$a_{2,33} = -\frac{3}{2}a_{3,222} - \frac{9}{2}g_1 - 3g_2 - \frac{9}{2}g_5 - 2g_6, \tag{93}$$

$$b_{1,1} = -B_{1,1}^2, \quad B_{1,1}^2 \geq 0, \tag{94}$$

$$b_{1,2} = B_{1,2}^2 \geq 0, \quad (95)$$

$$b_{2,12} = -\frac{1}{2}b_{1,3} + \zeta_1^2, \quad \zeta_1^2 \geq 0, \quad (96)$$

$$b_{2,23} = -\frac{3}{4}b_{3,122} - \frac{9}{8}(h_3 + h_4) + \zeta_3^2, \quad \zeta_3^2 \geq 0, \quad (97)$$

$$b_{2,13} = -\frac{1}{3}b_{2,22} - \frac{1}{2}b_{3,112} + \zeta_2^2, \quad \zeta_2^2 \geq 0. \quad (98)$$

It then turns out that these are sufficient conditions for making also Z_1 never vanish.

C. Positive μ^2 for $\forall X(t)$

In this process we have assumed that μ^2 is positive. However, we have to make sure it is. In fact, we find that on using the previous constraints, the squared mass for the tensor modes can be rewritten as

$$\begin{aligned} \mu^2 = & -\frac{1}{2}m^2B_{1,1}^2rX^3 - m^2(rB_{1,2}^2 + A_{1,2}^2)X^2 - \frac{1}{2}m^2A_{1,1}^2X \\ & + 6m^2r\zeta_2^2 + \frac{12}{X}m^2r\zeta_3^2 + \frac{54}{5X^2}m^2r\zeta_4^2, \end{aligned} \quad (99)$$

where

$$b_{2,33} = \zeta_4^2 - \frac{3}{2}b_{3,222} - \frac{9}{2}(h_1 + h_5) - 3h_2 - 2h_6, \quad (100)$$

so that we also need to impose

$$B_{1,1} = 0, \quad (101)$$

$$B_{1,2} = 0, \quad (102)$$

$$A_{1,2} = 0, \quad (103)$$

$$A_{1,1} = 0, \quad (104)$$

$$\zeta_4^2 \geq 0, \quad (105)$$

or

$$\mu^2 = 6m^2r \left(\zeta_2^2 + \frac{2}{X}\zeta_3^2 + \frac{9}{5X^2}\zeta_4^2 \right). \quad (106)$$

D. Finiteness of ISW effect

In the following, we show that the phenomenological criteria so far are sufficient to guarantee the finiteness of the ISW effect. For this purpose we use the equations of motion for scalar perturbations derived in Sec. IV C.

Since we are interested in the behavior of dust at late times, we will consider only one single pressureless fluid (modeling baryon and dark matter components). This leads to having effectively only one kind of matter component, for which the equations of motion reduce to

$$E_\theta = \dot{\delta}_m + \theta_m - 3\dot{\phi} = 0, \quad (107)$$

$$E_{\delta\rho/\rho} = \dot{\theta}_m + aH\theta_m - k^2\psi = 0. \quad (108)$$

Also we consider the subset of the extended theories which satisfy the conditions $\Xi_1 = 0 = \Xi_2$, and in particular the model and the constraints we have found in the previous section leading some coefficient to vanish, e.g. $S_{12} = 0$, as to have

$$E_E = S_1\dot{\phi} + S_2\phi + S_3\psi + S_{10}\theta_m = 0. \quad (109)$$

Instead, Eq. (76) reduces to

$$E_{E\zeta} = T_1\delta\lambda_V + T_2\phi + T_3\psi + T_4\delta_m + T_6\theta_m = 0, \quad (110)$$

which, as done before, can be used then to define $\delta\lambda_V$ in terms of the other variables. Finally the equation $E_{\delta\lambda}$ simplifies to

$$E_{\delta\lambda} = U_1\phi + U_2\delta_m + U_4\theta_m = 0. \quad (111)$$

Now, on taking the time derivative of Eq. (111), we have

$$\begin{aligned} \dot{E}_{\delta\lambda} = & (U_1 + 3U_2)\dot{\phi} + k^2U_4\psi + \dot{U}_1\phi + \dot{U}_2\delta_m \\ & + (\dot{U}_4 - U_2 - aHU_4)\theta_m = 0, \end{aligned}$$

where we have replaced the time derivative of the matter fields by using their own equations of motion. Then we can build up the following combination of equations of motion

$$\begin{aligned} E_S \equiv & (U_1 + 3U_2)E_E - S_1\dot{E}_{\delta\lambda} = [(U_1 + 3U_2)S_3 \\ & - k^2U_4S_1]\psi + [(U_1 + 3U_2)S_2 - S_1\dot{U}_1]\phi \\ & + [(U_1 + 3U_2)S_{10} - (\dot{U}_4 - U_2 - aHU_4)S_1]\theta_m \\ & - S_1\dot{U}_2\delta_m = 0. \end{aligned} \quad (112)$$

From this last equation, $E_S = 0$, we find

$$\psi = F_\psi(\phi, \theta_m, \delta_m). \quad (113)$$

On substituting this expression into $E_E = 0$, we can solve this equation for $\dot{\phi}$ as

$$\dot{\phi} = F_\dot{\phi}(\phi, \theta_m, \delta_m). \quad (114)$$

Then on replacing $\dot{\phi}$ in $E_\theta = 0$, we can solve it in terms of θ_m , finding

$$\theta_m = F_\theta(\phi, \delta_m, \dot{\delta}_m), \quad (115)$$

from which we also obtain

$$\psi = G_\psi(\phi, \delta_m, \dot{\delta}_m), \quad (116)$$

$$\dot{\phi} = G_\dot{\phi}(\phi, \delta_m, \dot{\delta}_m), \quad (117)$$

$$\dot{\theta}_m = F_{\dot{\theta}}(\phi, \delta_m, \dot{\delta}_m, \ddot{\delta}_m). \quad (118)$$

Then on substituting these expressions in $E_{\delta\rho/\rho} = 0$, we can solve it for ϕ as in

$$\phi = F_{\phi}(\delta_m, \dot{\delta}_m, \ddot{\delta}_m), \quad (119)$$

which in turn can be used to set

$$\psi = I_{\psi}(\delta_m, \dot{\delta}_m, \ddot{\delta}_m), \quad (120)$$

$$\theta_m = H_{\phi}(\delta_m, \dot{\delta}_m, \ddot{\delta}_m). \quad (121)$$

Finally we can substitute these last expressions for ϕ and θ_m into Eq. (111), $E_{\delta\lambda} = 0$, in order to find a closed differential equation for δ_m of the kind

$$\ddot{\delta}_m + A\dot{\delta}_m + B\delta_m = 0. \quad (122)$$

Once more, the reason why we can close the dynamical equation of motion for δ_m is that the theory does not add any new propagating degree of freedom in the scalar sector. In the high- k regime the previous equation reduces to

$$\ddot{\delta}_m + aH\dot{\delta}_m - \frac{3G_{\text{eff}}}{2G_N}\Omega_m a^2 H^2 \delta_m = 0, \quad (123)$$

where $\Omega_m = \frac{\rho_m}{3M_{\text{pl}}^2 H^2}$ and the concrete expression for G_{eff}/G_N is shown in Appendix D. This differential equation for δ_m can be used to replace $\ddot{\delta}_m$ in terms of $\delta_m, \dot{\delta}_m$, so that any scalar perturbation field becomes a function of $\delta_m, \dot{\delta}_m$ only. The result in this section for G_{eff}/G_N , following a different method, agrees with the one of the previous sections, as expected.⁸

At this point we have found that all the fields (except for δ_m itself) can be written as linear combinations of δ_m and $\dot{\delta}_m$. In particular, we can find the following combination, whose time derivative affects the ISW effect, namely

$$\psi_{\text{ISW}} \equiv \phi + \psi. \quad (124)$$

We find that in the high- k regime we have

$$\psi_{\text{ISW}} = -\frac{3H_0^2 \Omega_{m0}}{k^2} \Sigma \frac{\delta}{a}, \quad (125)$$

where $\Sigma = \Sigma(t)$, $\lim_{m/H \rightarrow 0} \Sigma = 1$ and its denominator never vanishes for any dynamics of $X(t)$. The general expression for this model is written in Appendix D.

E. General subclass

Finally, on putting together all the phenomenological criteria, we find that the model can be rewritten as

$$\begin{aligned} F_1 = & c_4 + \left(\frac{2}{9}[\mathfrak{R}]^3 - [\mathfrak{R}][\mathfrak{R}^2] + [\mathfrak{R}^3]\right)a_{1,3} + \left(2[\mathfrak{R}][\mathfrak{R}^2] - \frac{10}{9}[\mathfrak{R}]^3\right)\xi^2 + \left(\frac{1}{9}[\mathfrak{R}]^4 + 2[\mathfrak{R}][\mathfrak{R}^3] - 3[\mathfrak{R}^2]^2\right)a_{2,13} \\ & + \left([\mathfrak{R}^2]^2[\mathfrak{R}] - \frac{1}{18}[\mathfrak{R}]^5 - \frac{3}{2}[\mathfrak{R}^2][\mathfrak{R}^3]\right)a_{3,122} + \left([\mathfrak{R}^2]^3 - \frac{[\mathfrak{R}]^6}{54} - \frac{3[\mathfrak{R}^3]^2}{2}\right)a_{3,222} + \left([\mathfrak{R}^2][\mathfrak{R}]^4 - \frac{5}{18}[\mathfrak{R}]^6 - \frac{9}{2}[\mathfrak{R}^3]^2\right)g_1 \\ & + \left([\mathfrak{R}^2]^2[\mathfrak{R}]^2 - \frac{2}{27}[\mathfrak{R}]^6 - 3[\mathfrak{R}^3]^2\right)g_2 + \left(\frac{(4[\mathfrak{R}]^3 - 9[\mathfrak{R}^3])[\mathfrak{R}^2]}{4} - \frac{[\mathfrak{R}]^5}{4}\right)g_3 + \left(\frac{(36[\mathfrak{R}]^2 - 81[\mathfrak{R}^2])[\mathfrak{R}^3]}{36} - \frac{[\mathfrak{R}]^5}{36}\right)g_4 \\ & - \frac{([\mathfrak{R}]^3 - 9[\mathfrak{R}^3])^2 g_5}{18} + \left([\mathfrak{R}^2][\mathfrak{R}^3][\mathfrak{R}] - \frac{1}{81}[\mathfrak{R}]^6 - 2[\mathfrak{R}^3]^2\right)g_6 - \frac{([\mathfrak{R}]^2 - 3[\mathfrak{R}^2])^2 a_{3,112}}{6}. \end{aligned} \quad (126)$$

$$\begin{aligned} F_2 = & \left(\frac{2}{9}[\mathcal{K}]^3 - [\mathcal{K}][\mathcal{K}^2] + [\mathcal{K}^3]\right)b_{1,3} + \left(-\frac{1}{27}[\mathcal{K}]^4 - \frac{2}{3}[\mathcal{K}][\mathcal{K}^3] + [\mathcal{K}^2]^2\right)b_{2,22} + \left(-\frac{10}{9}[\mathcal{K}]^3 + 2[\mathcal{K}][\mathcal{K}^2]\right)\xi_1^2 \\ & + \left(2[\mathcal{K}][\mathcal{K}^3] - \frac{4}{9}[\mathcal{K}]^4\right)\xi_2^2 + \left(2[\mathcal{K}^2][\mathcal{K}^3] - \frac{2[\mathcal{K}]^5}{15}\right)\xi_3^2 + \left([\mathcal{K}^3]^2 - \frac{[\mathcal{K}]^6}{45}\right)\xi_4^2 \\ & + \left([\mathcal{K}^2]^2[\mathcal{K}]^2 - \frac{2}{27}[\mathcal{K}]^6 - 3[\mathcal{K}^3]^2\right)h_2 + \left(\frac{(4[\mathcal{K}]^3 - 9[\mathcal{K}^3])[\mathcal{K}^2]}{4} - \frac{[\mathcal{K}]^5}{4}\right)h_3 + \left(\frac{(36[\mathcal{K}]^2 - 81[\mathcal{K}^2])[\mathcal{K}^3]}{36} - \frac{[\mathcal{K}]^5}{36}\right)h_4 \\ & + \left([\mathcal{K}^2]^2[\mathcal{K}] - \frac{1}{18}[\mathcal{K}]^5 - \frac{3}{2}[\mathcal{K}^2][\mathcal{K}^3]\right)b_{3,122} + \left([\mathcal{K}^2]^3 - \frac{[\mathcal{K}]^6}{54} - \frac{3[\mathcal{K}^3]^2}{2}\right)b_{3,222} + \left([\mathcal{K}^2][\mathcal{K}]^4 - \frac{5}{18}[\mathcal{K}]^6 - \frac{9}{2}[\mathcal{K}^3]^2\right)h_1 \\ & + \left([\mathcal{K}^2][\mathcal{K}]^2 - \frac{2}{9}[\mathcal{K}]^4 - [\mathcal{K}][\mathcal{K}^3]\right)b_{3,112} - \frac{([\mathcal{K}]^3 - 9[\mathcal{K}^3])^2 h_5}{18} + \left(-\frac{1}{81}[\mathcal{K}]^6 + [\mathcal{K}^2][\mathcal{K}^3][\mathcal{K}] - 2[\mathcal{K}^3]^2\right)h_6. \end{aligned} \quad (127)$$

⁸In particular, this result shows that $\delta\rho/\rho = \delta_m$ in the high- k regime. It can be proven that this same result holds also for another gauge invariant combination, the comoving matter energy density defined as $\delta_v = \delta\rho/\rho + 3Hv$, namely $\delta\rho/\rho = \delta_v$.

For this class of models we can see that

$$\frac{\dot{X}}{NH} = \frac{5\zeta_1^2 X^3 + 10\zeta_2^2 X^2 + 10\zeta_3^2 X + 6\zeta_4^2}{5X^4 \xi^2} r - X, \quad (128)$$

whose dynamics are always well defined. Note two things: (1) a Λ CDM profile, i.e. $X = X_0 = \text{constant}$, can always be given for the background if necessary, and (2) on giving $X(t)$, we find $r(t)$, or vice versa, on giving $r(t)$, one needs to solve an ODE in order to find $X(t)$. It is interesting to notice that the Friedmann equation can then be written as

$$3M_{\text{p}}^2 H^2 = \sum_I \rho_I + m^2 M_{\text{p}}^2 (c_4 - 6\xi^2 X^3), \quad (129)$$

$$2M_{\text{p}}^2 \frac{\dot{H}}{N} = -\sum_I (\rho_I + P_I) - 6M_{\text{p}}^2 m^2 \xi^2 \frac{\dot{X}}{NH} X^2, \quad (130)$$

which simplify considerably. In this case the equation of state parameter for the eMTMG component becomes

$$w_g \equiv \frac{P_g}{\rho_g} = \frac{30(X^3 \zeta_1^2 + 2\zeta_2^2 X^2 + 2\zeta_3^2 X + \frac{6}{3}\zeta_4^2) r - 5X^2 c_4}{5X^2 (c_4 - 6\xi^2 X^3)}, \quad (131)$$

$$\rho_g \equiv m^2 M_{\text{p}}^2 (c_4 - 6\xi^2 X^3). \quad (132)$$

Also, as already mentioned, the general expression for G_{eff}/G_N and Σ for this general subclass can be found in Appendix D.

F. Simple subclass

We have obtained the general subclass of models in Sec. VI E. On the other hand, since the observables, ρ_g , G_{eff}/G_N , Σ and μ_{GW}^2 depend only on $(c_4, \xi, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$ among parameters in (126)–(127), we can pick up a simple subclass as follows⁹

$$F_1 = c_4 + \left(2[\mathfrak{K}][\mathfrak{K}^2] - \frac{10}{9}[\mathfrak{K}]^3 \right) \xi^2, \quad (133)$$

$$F_2 = \left(2[\mathcal{K}][\mathcal{K}^2] - \frac{10}{9}[\mathcal{K}]^3 \right) \zeta_1^2 + \left(2[\mathcal{K}][\mathcal{K}^3] - \frac{4}{9}[\mathcal{K}]^4 \right) \zeta_2^2 \\ + \left(2[\mathcal{K}^2][\mathcal{K}^3] - \frac{2[\mathcal{K}]^5}{15} \right) \zeta_3^2 + \left([\mathcal{K}^3]^2 - \frac{[\mathcal{K}]^6}{45} \right) \zeta_4^2, \quad (134)$$

⁹As we will see in Appendix C, the existence of the self-accelerating branch requires that $F_{1, [\mathfrak{K}]} + 2XF_{1, [\mathfrak{K}^2]} + 3X^2 F_{1, [\mathfrak{K}^3]} = 0$, which for this subclass leads to imposing $12\xi^2 X^2 = 0$, a solution which requires $\xi = 0$. On the other hand, we also need to impose $X^2 F_{2, [\mathcal{K}]} + 2XF_{2, [\mathcal{K}^2]} + 3F_{2, [\mathcal{K}^3]} = 0$, which is solved by setting the condition $5\zeta_1^2 X^3 + 10\zeta_2^2 X^2 + 10\zeta_3^2 X + 6\zeta_4^2 = 0$. In this case then $G_{\text{eff}}/G_N = 1$, and the phenomenology reduces to the one of Λ CDM, except, in general, for a nonzero value of the graviton mass.

which has six free parameters instead of four for MTMG (or dRGT model). This subset, at least on FLRW, is sufficiently general in the sense that it catches the behavior of a more general subclass of models presented in Sec. VI E. In general, from a purely theoretical approach, by using symmetries in the space of the functions (in fact, all of them would still belong to the minimal set of theories) we find it difficult to strongly restrict the allowed possibilities. Even then, still the final theory might not have a nice phenomenology. On the other hand, in this paper, we have shown that demanding for additional phenomenological considerations (all aimed as to give “a good fit to the data”) has proven a powerful way as to restrict the space of allowed functions for both F_1 and F_2 . The hope is that the real theory (fully determined theoretically) might look like one of these toy models.

In principle, one needs to be fitting all the free parameters of the model against the data, giving then predictions on the graviton mass. We will study the possibly interesting phenomenology for this theory in another separate paper.

VII. CONCLUSION

Nowadays, cosmology has reached an astonishingly high level of understanding of our universe due to more and more precise observations whose number also grows more and more. However, these data seem to give us a puzzling scenario regarding the dark sector of our universe. This is not only due to the long-standing problem of understanding the tiny value of the cosmological constant, for which a complete theoretical explanation is still unavailable. In fact, recent cosmological observations, as they reached a percent level of precision (at least for some of the experiments), show tensions and/or anomalies in the estimation of cosmological parameters such as the $z = 0$ Hubble expansion rate, H_0 , or the amplitude of matter fluctuations S_8 in the context of GR- Λ CDM. On the other hand, the fact that GR has shown to be fully compatible up to now with local gravity/astrophysical observations (including the experiments concerning gravitational waves) seems to leave little space to some deviations from it. Therefore this scenario leads to doubts on the experiments, or doubts on the analysis/interpretation of the data, or doubts on the theoretical model used to fit the same data. In particular, this third possibility opens up room for new (gravitational) physics and motivates, for instance, the study of various modified theories of gravity to address these same tensions.

We have tried, as further explained later, to modify gravity in this paper under the assumption that the graviton has a nonzero mass. Indeed, giving a mass to the graviton is a well-motivated scenario to consider. This issue has been attracting the attention of several physicists, since the first attempt by Fierz and Pauli, back in 1939 [21]. The full nonlinear realization of massive gravity was accomplished only very recently, which is now known as dRGT

theory [25]. Although the theory of dRGT is a valid theory for massive gravity, nonetheless it was proven that the cosmology of this theory was plagued with instabilities [27], as at least one (out of the five degrees of freedom) is a ghost (whose mass is in general below the cutoff of the theory). In [29] a new theory of massive gravity which was constructed as not to have the unstable mode of dRGT introduced. This theory, called “minimal theory of massive gravity” (MTMG), is said to be minimal in the sense that, it does not propagate any degrees of freedom other than the gravitational waves, which, on the other hand, are massive. This theory shows interesting phenomenology as it can lower the value of $f\sigma_8$ [45], since pressureless fluids can feel weaker gravity, as the effective gravitational constant is lower than the Newton constant, i.e. $G_{\text{eff}} < G_N$.

However, this modification of G_{eff} in MTMG consists of being a function of time with a pole at $\mu^2/H_\infty^2 = 2$, with μ^2 being the squared mass of the tensor modes in the theory and H_∞ being the value of the Hubble expansion rate at which $|G_{\text{eff}}| \rightarrow \infty$, where the background is nonetheless well defined and equal to Λ CDM. Then it is clear that the theory breaks down (its description as a low-energy effective theory), for $H_\infty > H_0$, i.e. $\mu^2 \geq 2H_0^2$, see e.g. [37,45]. However, if $\mu^2 < 0$ the pole is never encountered, and G_{eff} remains a smooth function at all times. The price to pay for this (in MTMG) is that the gravitational waves are tachyon fields, possessing a negative, but tiny-squared mass. This would lead to a tachyonic instability for them which is only effective for graviton-kinetic-energy of order H_0^2 (not visible at astrophysical scales) and a time of instability of order H_0^{-1} . Therefore in MTMG, either we live with tachyonic gravitational waves, or we have to avoid real (and larger than H_0) values for μ . This phenomena do limit the phenomenological possibilities of the normal branch of the original MTMG.

In this work, we extend the MTMG theory, motivated by the previous phenomenological behavior, as to remove the negative-squared mass behavior and, at the same time, any poles in G_{eff} . We impose these properties to be valid at any time and for any background dynamics. By doing this also other observables, such as the ISW field, will have a smooth evolution. In order to define the eMTMG, we first realize that the original MTMG was built as to have the same cosmological background as dRGT. Then, we allow the new class of theories to have a general graviton mass term and not only the MTMG/dRGT-like one. Afterwards, in order to have a theory with only tensor degrees of freedom in the gravity sector, we implement new constraints as to remove the unstable modes (already present in dRGT). Now the eMTMG leads to a mass term which consists of two functions: a function F_1 of $[\mathfrak{R}]$, $[\mathfrak{R}^2]$, $[\mathfrak{R}^3]$; and another function F_2 of $[\mathcal{K}]$, $[\mathcal{K}^2]$, $[\mathcal{K}^3]$ (where $[\mathfrak{R}]$, $\dots[\mathcal{K}]$, \dots depend on the three-dimensional metric γ_{ij} and a fiducial metric $\tilde{\gamma}_{ij}$). This choice naturally vastly expands the phenomenology of MTMG in general. After investigating

the background equations of motion, we have studied the tensor mode perturbations, and found that the two polarizations of the gravitational waves acquire a nontrivial mass as expected.

Later on, we impose the conditions mentioned above for $\mu^2 \geq 0$ and finiteness of G_{eff} . We have found that this model can lead to three possible different phenomenologies, which depend on two functions Ξ_1 and Ξ_2 which, in turn, depend on $F_{1,2}$ and their derivatives. In fact, we find that if $\Xi_1 \neq 0$, then $G_{\text{eff}} = G_N$ (evidently MTMG does not belong to this class). If instead $\Xi_1 = 0$ [or much smaller than $k^2/(a^2H^2)$] but $\Xi_2 \neq 0$, in general the speed of propagation for each matter field will be modified. Finally for the subclass of theories for which $\Xi_1 = 0 = \Xi_2$, matter component has the standard speed of propagation, whereas dust acquires a nontrivial G_{eff}/G_N . MTMG belongs to this last class. For this last class we proceed to impose the conditions $\mu^2 \geq 0$ and $G_{\text{eff}} < \infty$, and we give an explicit example which satisfies these constraints at all times for any background dynamics.

In this last case, the expression for G_{eff}/G_N is explicitly given in Appendix D. It is interesting to notice that the mass squared of the graviton could be vanishing, whereas $G_{\text{eff}} \neq G_N$. This is due to the fact that at linear level, the contributions to G_{eff} come from the would-be-unstable propagating scalar mode of dRGT which in this minimal theory is nondynamical and as such can be integrated out, leading though to nonstandard modifications to the coefficients of the linear perturbation equations of motion.

We have extended the study of finiteness to other linear perturbation observables as to see how their late time dynamics are affected. In particular, we have looked at the observable which describes the ISW-galaxy correlation effects. Indeed we find that imposing $\mu^2 \geq 0$ and $G_{\text{eff}} < \infty$ automatically leads to the absence of poles for such observables.

The result of this work is interesting since it provides a set of eMTMG which, like GR, leads to cosmological observables which are always well defined, no matter which dynamics the background might have. These requirements can turn out to be crucial in a world which has to deal with a weak gravity description of large-scale gravitational interactions. We think these minimal models could have an interesting phenomenology leading to new possibilities for a massive graviton to play a nontrivial role in our physical world.

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APPENDIX A: VARIATIONAL FORMULAS

We find it useful to write down some identities which are to be used when we find the equations of motion of the theory¹⁰

$$\delta\mathcal{K}^i{}_i = \frac{1}{2}\mathfrak{R}^i{}_j\tilde{\gamma}^{jl}\delta\gamma_{li}, \quad (\text{A1})$$

$$\partial_t\mathcal{K}^i{}_i = \frac{1}{2}\mathfrak{R}^i{}_j\dot{\tilde{\gamma}}^{jl}\gamma_{li} = -M\mathfrak{R}^i{}_j\bar{\zeta}^j{}_i, \quad (\text{A2})$$

$$\delta(\mathcal{K}^i{}_j\mathcal{K}^j{}_i) = \tilde{\gamma}^{ij}\delta\gamma_{ij}, \quad (\text{A3})$$

$$\partial_t(\mathcal{K}^i{}_j\mathcal{K}^j{}_i) = \dot{\tilde{\gamma}}^{il}\gamma_{li} = -2M\bar{\zeta}^i{}_j\tilde{\gamma}^{jl}\gamma_{li}, \quad (\text{A4})$$

$$\delta(\mathcal{K}^i{}_j\mathcal{K}^j{}_k\mathcal{K}^k{}_i) = \frac{3}{2}\mathcal{K}^i{}_j\tilde{\gamma}^{jl}\delta\gamma_{li}, \quad (\text{A5})$$

$$\partial_t(\mathcal{K}^i{}_j\mathcal{K}^j{}_k\mathcal{K}^k{}_i) = \frac{3}{2}\mathcal{K}^i{}_j\dot{\tilde{\gamma}}^{jl}\gamma_{li} = -3M\mathcal{K}^i{}_j\bar{\zeta}^j{}_i\tilde{\gamma}^{lk}\gamma_{ki}, \quad (\text{A6})$$

where a δ represents the variation with respect to a dynamical field, and ∂_t the time derivative of the explicitly time dependent fields.

Now let us turn our attention to the analogue properties of the other squared-root matrix, namely $\mathfrak{R}^i{}_j$. Then we find

$$\begin{aligned} \delta\mathfrak{R}^i{}_i &= \frac{1}{2}\mathcal{K}^i{}_j\delta\gamma^{jl}\tilde{\gamma}_{li} = -\frac{1}{2}\mathcal{K}^i{}_j\gamma^{kl}\tilde{\gamma}_{li}\gamma^{jm}\delta\gamma_{mk} \\ &= -\frac{1}{2}\mathfrak{R}^i{}_j\gamma^{jk}\delta\gamma_{ki}, \end{aligned} \quad (\text{A7})$$

$$\partial_t\mathfrak{R}^i{}_i = \frac{1}{2}\mathcal{K}^i{}_j\gamma^{jl}\dot{\tilde{\gamma}}_{il} = M\tilde{\gamma}_{ij}\mathcal{K}^i{}_l\gamma^{kl}\bar{\zeta}^j{}_k, \quad (\text{A8})$$

$$\delta(\mathfrak{R}^i{}_j\mathfrak{R}^j{}_i) = \delta\gamma^{ij}\tilde{\gamma}_{ij} = -\gamma^{ij}\tilde{\gamma}_{jl}\gamma^{lm}\delta\gamma_{mi}, \quad (\text{A9})$$

$$\partial_t(\mathfrak{R}^i{}_j\mathfrak{R}^j{}_i) = 2M\gamma^{ij}\tilde{\gamma}_{jl}\bar{\zeta}^i{}_i, \quad (\text{A10})$$

$$\delta(\mathfrak{R}^i{}_j\mathfrak{R}^j{}_k\mathfrak{R}^k{}_i) = \frac{3}{2}\mathfrak{R}^i{}_j\tilde{\gamma}_{li}\delta\gamma^{jl} = -\frac{3}{2}\mathfrak{R}^i{}_j\tilde{\gamma}_{li}\gamma^{ml}\gamma^{jk}\delta\gamma_{km}, \quad (\text{A11})$$

$$\partial_t(\mathfrak{R}^i{}_j\mathfrak{R}^j{}_k\mathfrak{R}^k{}_i) = \frac{3}{2}\mathfrak{R}^i{}_j\gamma^{jl}\dot{\tilde{\gamma}}_{li} = 3M\mathfrak{R}^i{}_j\gamma^{jl}\tilde{\gamma}_{lm}\bar{\zeta}^m{}_i. \quad (\text{A12})$$

¹⁰In the following we will make use of the identity $\delta\{\text{Tr}[\sqrt{X^n}]\} = \frac{n}{2}\text{Tr}[\sqrt{X^{n-2}}\delta X]$.

APPENDIX B: MTMG SUBCASE

In the theory of MTMG, a subclass of the eMTMG, we have for the precursor part of the Lagrangian the following structure

$$\begin{aligned} \mathcal{L}_{\text{MTMG}} &\ni \frac{m^2 M_{\text{P}}^2}{2} N [-c_1 \sqrt{\tilde{\gamma}} - c_2 \sqrt{\tilde{\gamma}}[\mathcal{K}] - c_3 \sqrt{\tilde{\gamma}}[\mathfrak{R}] - c_4 \sqrt{\tilde{\gamma}}] \\ &= -\frac{m^2 M_{\text{P}}^2}{2} N \sqrt{\tilde{\gamma}} \left[c_1 \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} + c_2 [\mathcal{K}] \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} + c_3 [\mathfrak{R}] + c_4 \right], \end{aligned} \quad (\text{B1})$$

where, by definition

$$\begin{aligned} \mathfrak{R}^a{}_b\mathfrak{R}^b{}_c &= \gamma^{ab}\tilde{\gamma}_{bc}, \\ [\mathcal{K}] &= [\mathfrak{R}^{-1}] \end{aligned}$$

so that

$$\det(\mathfrak{R})^2 = \frac{\tilde{\gamma}}{\gamma}, \quad \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} = \det(\mathfrak{R}), \quad (\text{B2})$$

supposing that $\det(\mathfrak{R}) > 0$. By using the Cayley-Hamilton (CH) theorem¹¹ we have

$$[\mathfrak{R}^3] - [\mathfrak{R}][\mathfrak{R}^2] + \frac{1}{2}([\mathfrak{R}]^2 - [\mathfrak{R}^2])[\mathfrak{R}] - 3\det(\mathfrak{R}) = 0, \quad (\text{B3})$$

so that

$$\det(\mathfrak{R}) = \frac{1}{3}[\mathfrak{R}^3] - \frac{1}{2}[\mathfrak{R}][\mathfrak{R}^2] + \frac{1}{6}[\mathfrak{R}]^3. \quad (\text{B4})$$

We also have from the CH theorem that:

$$[\mathfrak{R}^2] - [\mathfrak{R}]^2 + \frac{3}{2}([\mathfrak{R}]^2 - [\mathfrak{R}^2]) - \det(\mathfrak{R})[\mathcal{K}] = 0, \quad (\text{B5})$$

and

$$[\mathcal{K}]\det(\mathfrak{R}) = \frac{1}{2}([\mathfrak{R}]^2 - [\mathfrak{R}^2]). \quad (\text{B6})$$

Since

¹¹For a three-dimensional matrix, A , one has:

$$A^3 - \text{tr}(A)A^2 + \frac{1}{2}[(\text{tr}A)^2 - \text{tr}(A^2)]A - \det(A)I_3 = 0,$$

out of which we can take the trace or multiply it by A^{-1} to find new useful relations.

$$\begin{aligned} \mathcal{L}_{\text{MTMG}} \ni & -\frac{m^2 M_{\text{P}}^2}{2} N \sqrt{\gamma} [c_1 \det(\mathfrak{K}) + c_2 [\mathcal{K}] \det(\mathfrak{K}) \\ & + c_3 [\mathfrak{K}] + c_4], \end{aligned} \quad (\text{B7})$$

we have that for MTMG:

$$\begin{aligned} F_1^{\text{MTMG}} = & c_1 \left(\frac{1}{3} [\mathfrak{K}^3] - \frac{1}{2} [\mathfrak{K}] [\mathfrak{K}^2] + \frac{1}{6} [\mathfrak{K}^3] \right) \\ & + \frac{1}{2} c_2 ([\mathfrak{K}]^2 - [\mathfrak{K}^2]) + c_3 [\mathfrak{K}] + c_4. \end{aligned} \quad (\text{B8})$$

Along the same lines one finds

$$\begin{aligned} \mathcal{L}_{\text{MTMG}} \ni & \frac{m^2 M_{\text{P}}^2}{2} M [-c_1 [\mathcal{K}] \sqrt{\tilde{\gamma}} - \frac{1}{2} c_2 \sqrt{\tilde{\gamma}} ([\mathcal{K}]^2 - [\mathcal{K}^2]) \\ & - c_3 \sqrt{\tilde{\gamma}}] \\ = & -\frac{m^2 M_{\text{P}}^2}{2} M \sqrt{\tilde{\gamma}} \left[c_1 [\mathcal{K}] + \frac{1}{2} c_2 ([\mathcal{K}]^2 \right. \\ & \left. - [\mathcal{K}^2]) + c_3 \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\tilde{\gamma}}} \right], \end{aligned} \quad (\text{B9})$$

where

$$\mathcal{K}^a_b \mathcal{K}^b_c = \tilde{\gamma}^{ab} \gamma_{bc}, \quad (\text{B10})$$

so that

$$\det(\mathcal{K})^2 = \frac{\gamma}{\tilde{\gamma}}, \quad (\text{B11})$$

or

$$\frac{\sqrt{\gamma}}{\sqrt{\tilde{\gamma}}} = \det(\mathcal{K}), \quad (\text{B12})$$

supposing that $\det(\mathcal{K}) > 0$. Then on using once more the CH theorem one finds

$$\det(\mathcal{K}) = \frac{1}{3} [\mathcal{K}^3] - \frac{1}{2} [\mathcal{K}] [\mathcal{K}^2] + \frac{1}{6} [\mathcal{K}^3], \quad (\text{B13})$$

and, finally, that

$$\begin{aligned} F_2^{\text{MTMG}} = & c_1 [\mathcal{K}] + \frac{1}{2} c_2 ([\mathcal{K}]^2 - [\mathcal{K}^2]) \\ & + c_3 \left(\frac{1}{3} [\mathcal{K}^3] - \frac{1}{2} [\mathcal{K}] [\mathcal{K}^2] + \frac{1}{6} [\mathcal{K}^3] \right). \end{aligned} \quad (\text{B14})$$

APPENDIX C: SELF-ACCELERATING BRANCH

Let us once more consider the nontrivial constraint equation Eq. (45), that we rewrite here for later convenience

$$\begin{aligned} H \frac{M}{N} (X^2 F_{2, [\mathcal{K}]} + 2X F_{2, [\mathcal{K}^2]} + 3F_{2, [\mathcal{K}^3]}) \\ = \left(\frac{\dot{X}}{N} + HX \right) (F_{1, [\mathfrak{K}]} + 2X F_{1, [\mathfrak{K}^2]} + 3X^2 F_{1, [\mathfrak{K}^3]}). \end{aligned} \quad (\text{C1})$$

We can define a self-accelerating branch for these extended minimal models, as the solution of this constraint which does not fix the ratio M/N . For this to happen we require

$$X^2 F_{2, [\mathcal{K}]} + 2X F_{2, [\mathcal{K}^2]} + 3F_{2, [\mathcal{K}^3]} = 0, \quad (\text{C2})$$

which is an algebraic equation for X . In particular, this equation implies that $X = X_0 = \text{constant}$. Since, from our assumptions $X_0 \neq 0$, in general, Eq. (C1) also leads to

$$F_{1, [\mathfrak{K}]} + 2X F_{1, [\mathfrak{K}^2]} + 3X^2 F_{1, [\mathfrak{K}^3]} = 0. \quad (\text{C3})$$

Vice versa, if we assume Eq. (C3) holding true, then since we assume that HM/N does not vanish, we are left to impose that also Eq. (C2) needs to be satisfied. Then both Eqs. (C2) and (C3) must hold at the same time, meaning that X_0 has to be a solution for both these equations. In this case, we will name this possibility as the self-accelerating solution. This solution might not exist for all possible $F_{1,2}$ functions, but there will be subclass of theories admitting its presence. In particular MTMG is one of them.

For the self-accelerating branch, as defined here, we find that both the background and the scalar/vector linear perturbation equations behave exactly as in General Relativity, and in particular, $\frac{1}{2} M_{\text{P}}^2 m^2 F_1$ reduce to an effective cosmological constant contribution to the total matter sector. In summary, for this solution, all the phenomenology (up to linear perturbations in cosmology) coincide with GR except for the tensor modes which acquire a nonzero mass (possibly time dependent).

APPENDIX D: FULL EXPRESSION OF G_{eff}/G_N

In the following we give a full expression for G_{eff}/G_N which can be written as

$$\begin{aligned}
\frac{G_{\text{eff}}}{G_N} = \frac{1}{\Delta} & \left\{ -6750\xi^2 X^3 \left(X^3 \zeta_1^2 + 2X^2 \zeta_2^2 + 2X \zeta_3^2 + \frac{6}{5} \zeta_4^2 \right) \left(X^5 \xi^2 - X^3 r \zeta_1^2 - 2X^2 r \zeta_2^2 - 2X r \zeta_3^2 - \frac{6}{5} r \zeta_4^2 \right) \right. \\
& \times \left(X^3 \zeta_1^2 + \frac{8}{3} X^2 \zeta_2^2 + \frac{10}{3} X \zeta_3^2 + \frac{12}{5} \zeta_4^2 \right) m^4 + 3375 H^2 m^2 \left[\xi^2 \zeta_1^4 \left(\Omega_m + \frac{2}{3} \right) X^{11} + \frac{14 \zeta_2^2 \xi^2 \left(\Omega_m + \frac{6}{7} \right) \zeta_1^2 X^{10}}{3} \right. \\
& + \frac{16 \left[\left(\Omega_m + \frac{4}{3} \right) \zeta_2^4 + \zeta_1^2 \zeta_3^2 \left(\Omega_m + \frac{5}{6} \right) \right] \xi^2 X^9}{3} + \left(\left(12 \left(\Omega_m + \frac{14}{9} \right) \zeta_3^2 \zeta_2^2 + \frac{18 \zeta_4^2 \left(\Omega_m + \frac{2}{3} \right) \zeta_1^2}{5} \right) \xi^2 - \frac{8 r \zeta_1^4 \zeta_2^2}{9} \right) X^8 \\
& + \left(\left(8 \zeta_4^2 \left(\Omega_m + \frac{8}{5} \right) \zeta_2^2 + \frac{20 \zeta_3^4 \left(\Omega_m + 2 \right)}{3} \right) \xi^2 - \frac{8 r \zeta_1^2 \left(\zeta_1^2 \zeta_3^2 + 6 \zeta_2^4 \right)}{9} \right) X^7 \\
& + \left(\frac{44 \left(\Omega_m + \frac{74}{33} \right) \zeta_4^2 \xi^2 \zeta_3^2}{5} - \frac{128 r \zeta_2^2 \left(\zeta_1^2 \zeta_3^2 + \frac{\zeta_4^2}{2} \right)}{9} \right) X^6 - \frac{144 r \left(\zeta_1^2 \zeta_4^4 + \frac{260}{27} \zeta_2^2 \zeta_3^2 \zeta_4^2 + \frac{250}{81} \zeta_3^6 \right) X^3}{25} \\
& + \left(\frac{72 \zeta_4^4 \xi^2 \left(\Omega_m + \frac{8}{3} \right)}{25} - \frac{128 \left(\zeta_1^2 \zeta_2^2 \zeta_4^2 + \frac{5}{4} \zeta_1^2 \zeta_3^4 + \frac{10}{3} \zeta_2^4 \zeta_3^2 \right) r}{15} \right) X^5 - \frac{224 r \left(\zeta_1^2 \zeta_3^2 \zeta_4^2 + \frac{9}{7} \zeta_2^4 \zeta_4^2 + \frac{55}{21} \zeta_2^2 \zeta_3^4 \right) X^4}{15} \\
& - \frac{512 r \left(\zeta_2^2 \zeta_4^2 + \frac{15 \zeta_3^4}{8} \right) \zeta_4^2 X^2}{25} - \frac{704 X r \zeta_3^2 \zeta_4^4}{25} - \frac{864 r \zeta_4^6}{125} \left. \right\} \\
& + 10 H^4 X^2 \left(15 X^3 \zeta_1^2 + 40 X^2 \zeta_2^2 + 50 X \zeta_3^2 + 36 \zeta_4^2 \right)^2, \tag{D1}
\end{aligned}$$

$$\Delta \equiv 2250 X^2 \left(X^3 \xi^2 \left(X^3 \zeta_1^2 + 2X^2 \zeta_2^2 + 2X \zeta_3^2 + \frac{6}{5} \zeta_4^2 \right) m^2 + \frac{H^2 \left(15 X^3 \zeta_1^2 + 40 X^2 \zeta_2^2 + 50 X \zeta_3^2 + 36 \zeta_4^2 \right)}{15} \right)^2, \tag{D2}$$

where we can explicitly see that the denominator Δ never vanishes. We give in the following the general expression for Σ , defined in Eq. (125), which can be written as

$$\Sigma = \frac{1}{2} \frac{G_{\text{eff}}}{G_N} + \frac{H^2 \left(15 X^3 \zeta_1^2 + 40 X^2 \zeta_2^2 + 50 X \zeta_3^2 + 36 \zeta_4^2 \right)}{30 X^3 \xi^2 \left(X^3 \zeta_1^2 + 2X^2 \zeta_2^2 + 2X \zeta_3^2 + \frac{6}{5} \zeta_4^2 \right) m^2 + 2H^2 \left(15 X^3 \zeta_1^2 + 40 X^2 \zeta_2^2 + 50 X \zeta_3^2 + 36 \zeta_4^2 \right)}, \tag{D3}$$

which never blows up to infinity for any dynamics of $X(t)$, and still reduces to unity when $m/H \rightarrow 0$, i.e. at early times.

APPENDIX E: CASE WITH MASSLESS GRAVITATIONAL WAVES

For the special symmetric¹² model $\zeta_2 = \zeta_3 = \zeta_4 = 0$, the tensor modes become effectively massless on the FLRW background, for any $X(t)$. It is interesting to note that even if μ^2 vanishes, still we might have nontrivial dynamics in the scalar sector as

$$\frac{G_{\text{eff}}}{G_N} = \frac{1 - 3\xi^4 Y^2 X^6 + 3X^3 (XY r \zeta_1^2 + \frac{\Omega_m}{2} + \frac{1}{3}) Y \xi^2}{(Y \xi^2 X^3 + 1)^2}, \quad Y \equiv \frac{m^2}{H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3M_{\text{p}}^2 H^2}, \quad \text{for } \zeta_2 = \zeta_3 = \zeta_4 = 0, \tag{E1}$$

which can be still less than unity (but positive) today.

The background function Σ defined in (125) and shown in Appendix D reduces to

$$\Sigma = \frac{2 - 3\xi^4 Y^2 X^6 + 3X^3 Y (XY r \zeta_1^2 + \frac{\Omega_m}{2} + \frac{2}{3}) \xi^2}{2(Y \xi^2 X^3 + 1)^2}, \quad Y \equiv \frac{m^2}{H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3M_{\text{p}}^2 H^2}, \quad \text{for } \zeta_2 = \zeta_3 = \zeta_4 = 0. \tag{E2}$$

Here, we have mentioned this choice for the parameters only as to give already a nontrivial example despite vanishing mass for gravitational waves. In the rest of the present paper we shall mainly consider the more general cases, i.e. those shown in Sec. VI E or VI F, with massive gravitational waves.

¹²We name it ‘‘symmetric’’ as in this case Eqs. (133) and (134) mirror each other (a cosmological constant in the fiducial sector can always be added without modifying any bit of the theory).

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