Mid-Semester Examination

Ph. D. Coursework, NAS-MUNA

Symmetries & Lie Algebra in Physics

(NWTP 702)

Instructor: Kumar Abhinav

Date: March 26, 2025

Time 09:00 - 11:00 hrs

Total time: 120 minutes | Semester 2/2024 | Total marks: 20

Instructions

I. Attempt **any 2** out of the given 5 questions.

II. Use **ONLY** your class notebook(s).

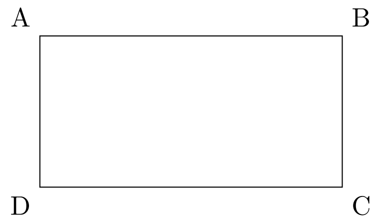
III. Use either blue or black ink.

IV. Try to submit on-time.

V. Individual marks are given in parentheses.

Questions:

1. Consider the rectangle:



in 2-dimensional space.

   a) Identify the symmetry group elements and construct the multiplication table. [5]

   b) Identify the subgroup(s) and construct all independent cosets. [1+2]

   c) How may irreducible representations are there and why? [2]

2. Consider the group of 2×2 real matrices $\{M\}$ that keep the combination $x^2 - y^2$ fixed in the $x-y$-space:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}, \quad X^2 - Y^2 = x^2 - y^2,$$

and also $|M| = 1$.

   a) How many independent real parameters are there? [2]

   b) Construct a matrix representation of this group. [5]

   c) Find out the generator matrices of the corresponding Lie group. [3]

3. Consider the group $\mathbb{Z}_3 = \left\{e, R, R^2\right\}$ such that $R \circ_G R = R^2$ and $R^3 = e$.

   a) How many irreducible representations are there for this group? [2]

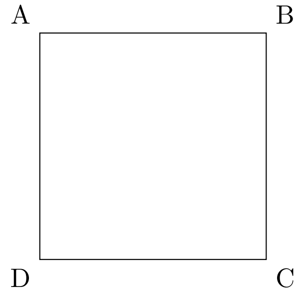
   b) Construct a non-trivial irreducible representation for this group. [3]

c) Construct the character table for $\mathbb{Z}_3$ using the relations,

$$\sum_{\alpha} K_\alpha \left(\chi^{(a)}(\alpha)\right)^* \chi^{(b)}(\alpha) = N\delta_{ab}, \quad \sum_a n_a^2 = N,$$

wherein α represents different conjugacy classes of the group with number of elements K_α and character $\chi(\alpha)$, whereas a, b stand for irreducible representations of the group. N is the group order and n_a is the dimension of representation a. [5]

4. Consider the group of rotation of the square,



in 2-dimensional space $\mathbb{Z}_4 = \left\{e, R, R^2, R^3\right\}$.

a) Construct the Young Tableaux and explain why they are appropriate for this group. [4]

b) Write down a non-trivial irreducible representation. [4]

c) How many irreducible representations are there and why? [2]

5. a) Show that the operator,

$$P_a = \frac{n_a}{N} \sum_g \left(\chi^{(a)}(g)\right)^* D(g),$$

is a projection operator onto the subspace of irreducible representation a of dimension n_a with character $\chi^{(a)}(g)$ for an arbitrary representation $D(g)$ of the group with order N. [5]

b) Given the 3-dimensional reducible representation,

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(\alpha) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(\beta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(B) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D(C) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

of S_3 show that $P_1 = 0$ where the representation $a = 1$ corresponds to the irreducible representation,

$$D^{(1)}(e, \alpha, \beta) = 1 = -D^{(1)}(A, B, C).$$ [5]

Best wishes