Centre for Theoretical Physics & Natural Philosophy
"Nakhonsawan Studiorum for Advanced Studies" - NAS

Assignment 1

Ph. D. Coursework, NAS-MUNA

Integrable System

(NWTP 601)

Instructor: Kumar Abhinav

Date: September 2nd, 2025

Due on September 16th, 2025 | Semester 1/2025 | Total marks: 30

Instructions

I. Attempt any 3 of the 4 questions.

II. Submit answers both in hard and soft (scanned) copies. Do not waste time by typing it out.

III. Use either blue or black ink.

IV. Delay in submission may reduce marks.

V. Individual marks are given in parentheses.

1. Consider N particles constrained to move along a circle of radius R in $\mathbb{R}^3$ (Do not consider the circle as a physical object, only the particles are.).

    a) What is the total number of degrees of freedom of this system and why? [1+1]

    b) Write down the Lagrangian of this system and obtain the expressions for the corresponding generalized momenta. [1+2]

    c) Obtain the Hamiltonian for this system through Legendre transformation. [2]

    d) Suppose the circle can now rotate about one of its diameters. What will be the number of degrees of freedom of this system now? Justify your answer. [1+2]

2. Consider a simple pendulum with point mass m and inextensible length l suspended from a rigid support that oscillates by a macroscopic angle $\theta > 1$.

    a) Obtain the Lagrangian and thereby the equations of motion for this system. [4+2]

    b) Obtain the the phase-space trajectory for fixed energy and plot it accurately (choose appropriate values of he system parameters at your convenience). Explain why you observe such a trajectory. [3+1]

3. Consider the isotropic 2-dimensional Harmonic oscillator:

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{K}{2}\left(x^2 + y^2\right), \quad p_x = m\dot{x}, \quad p_y = m\dot{y}.$$

    a) Check if the angular momentum $L = xp_y - yp_x$ is conserved or not using the Poisson bracket of the system. [3]

    b) Consider the coordinate transformation,

$$(x, y) \to (r, \theta) \quad \text{where} \quad r^2 = x^2 + y^2 \quad \& \quad \theta = \arctan\left(\frac{y}{x}\right),$$

    with the subsequent change $(p_x, p_y) = (p_r, p_\theta)$ of generalized momenta. Check whether it is a canonical transformation or not. [5]

c) Consider the functions,

$$u(x, y, p_x, p_y) = p_x^2 + p_y^2 = U(r, \theta, p_r, p_\theta), \quad v(x, y, p_x, p_y) = x^2 + y^2 = V(r, \theta, p_r, p_\theta).$$

Show that,

$$\{u, v\}_{x,p_x,y,p_y} = \{U, V\}_{r,p_r,\theta,p_\theta},$$

where the suffixes signify Poisson brackets in the particular phase-space basis. [2]

4. a) Why the Poisson bracket is important for a Hamiltonian system? [2]

b) Show that if you have two conserved quantities $u(q, p)$ and $v(q, p)$ in a Hamiltonian system then you can construct an infinite hierarchy of conserved quantities (mutually independent or not). [5]

c) In covariant notation for the canonical variables, show that,

$$\{A(y), B(y)\} = \partial_\mu A \{y^\mu, y^\nu\} \partial_\nu B,$$

where A and B are two dynamical quantities of the system. [3]

Best wishes